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9 Model-Free Prediction & Control (MC and TD Learning, Sarsa and Q-Learning)

Model-Free Reinforcement Learning

Last Module (8):

- Integrating learning and planning
- Use planning to construct a value function or policy

This Module (9):

- Model-free prediction and control
- Prediction: *Optimise* the value function of an unknown MDP
- Control: Learn model directly from *experience*

Monte-Carlo Learning

Monte-Carlo Reinforcement Learning

MC methods learn directly from episodes of experience

- MC is model-free: no knowledge of MDP transitions / rewards

MC learns from complete episodes

- No bootstrapping, as we will see later

MC uses the simplest possible idea of looking at sample returns: *value = mean return*

- Caveat: can only apply to *episodic* MDPs, i.e. all episodes must terminate

Monte-Carlo Policy Evaluation

- Goal: learn v_π from episodes of *experience* under policy π

$$S_1, A_1, R_2, \ldots, S_k \sim \pi$$

- Recall that the *return* is the total discounted reward:

$$G_t = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{T-1} R_T$$

- Recall that the value function is the expected return:

$$v_\pi(s) = \mathbb{E}_\pi \left[G_t \mid S_t = s \right]$$

- Monte-Carlo policy evaluation uses *empirical mean* return instead of *expected* return
- Computes empirical mean from the point t onwards using as many samples as we can
- Will be different for every time step

First-Visit Monte-Carlo Policy Evaluation

To evaluate state, s:

- On the first time-step, t that state, s, is visited in an episode:
  - Increment counter $N(s) \leftarrow N(s) + 1$
  - Increment total return $S(s) \leftarrow S(s) + G_t$
- Estimate:

$$V(s) = \frac{S(s)}{N(s)}$$

By Law of Large Numbers, $V(s) \to v_\pi(s)$ as $N(s) \to \infty$

- Only requirement is we somehow visit all of these states

The **Central Limit Theorem** tells us how quickly it approaches the mean

- The variance (mean squared error) of estimator reduces with $\frac{1}{N}$
- i.e. rate is independent of size of state space, $|s|$
- speed depends on how many episodes/visits reach s (coverage probabilities).

Every-Visit Monte-Carlo Policy Evaluation

To evaluate state, s:

- Every time-step, t, that state, s, is visited in an episode:
  - Increment counter $N(s) \leftarrow N(s) + 1$
  - Increment total return $S(s) \leftarrow S(s) + G_t$
- Estimate:

$$V(s) = \frac{S(s)}{N(s)}$$

Again, $V(s) \to v_\pi(s)$ as $N(s) \to \infty$.

First-Visit versus Every-Visit Monte-Carlo Policy Evaluation?

Every-Visit Advantages:

- Especially good when episodes are short or when states are *rarely* visited — no sample gets "wasted."
- i.e. uses more of the data collected, often faster convergence in practice.

First-visit Advantages:

- Useful when episodes are long and states repeated many times.
- i.e. avoids dependence between multiple visits to the same state in one episode.

Blackjack Example

States (~ 200):

- Current sum of cards ($12 - 21$)
- Dealer's showing card (Ace-10)
- Whether you have a *usable* ace - (can be counted as 1 *or* 11 without *busting* > 21) (yes/no)

Actions:

- Stand: stop receiving cards (and terminate)
- Hit: take another card (no replacement)

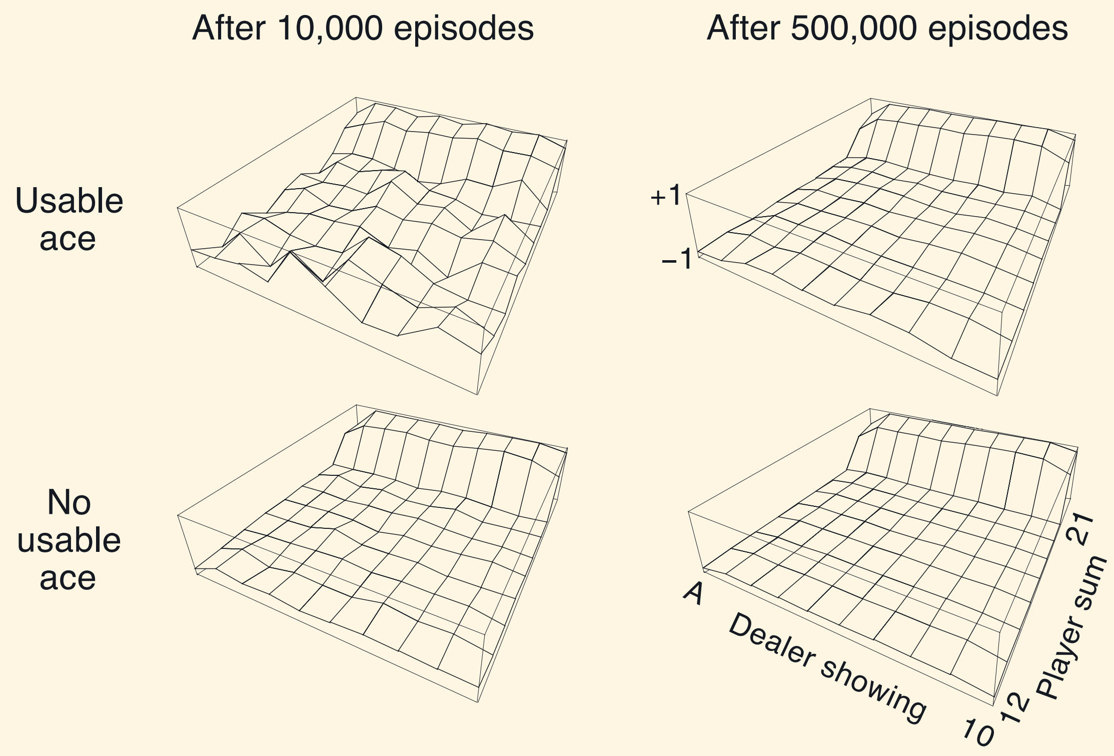
Transitions:

- You are automatically hit if your sum < 12

Rewards:

- For stand: $+1$ if your sum $>$ dealer's; 0 if equal; -1 if less
- For hit: -1 if your sum > 21 (and terminate); 0 otherwise

Blackjack Value Function after Monte-Carlo Learning



Policy: *stand* if sum of cards ≥ 20, otherwise *hit*

- Learning value function directly from experience
- Usable ace value is noisier because the state is *rarer*

Key point: once we have learned the value function from experience,

- we can *evaluate* actions for making the best decision for optimising a *policy* as we will see later

Incremental Mean (Refresher)

The mean $\mu_1, \mu_2, \cdots$ of a sequence $x_1, x_2, \ldots$ can be computed incrementally,

$$\begin{aligned}
\mu_k &= \frac{1}{k}\sum_{j=1}^{k} x_j \\
&= \frac{1}{k}\left(x_k + \sum_{j=1}^{k-1} x_j\right) \\
&= \frac{1}{k}(x_k + (k-1)\mu_{k-1}) \\
&= \mu_{k-1} + \frac{1}{k}(x_k - \mu_{k-1})
\end{aligned}$$

- μ_{k-1} is the previous mean: *predicts* what think value will be
- x_k is the new value
- Incrementally *corrects* mean $\frac{1}{k}$ in direction of error $x_k - \mu_{k-1}$

Incremental Monte-Carlo Updates (Same idea)

Update $V(s)$ incrementally after each episode $S_1, A_1, R_2, \ldots, S_T$:

- For each state S_t with return G_t:

$$N(S_t) \leftarrow N(S_t) + 1$$

$$V(S_t) \leftarrow V(S_t) + \frac{1}{N(S_t)} \left(G_t - V(S_t) \right)$$

In non-stationary problems, it can be useful to track a *running mean* by forgetting old episodes using a *constant* step size α:

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t - V(S_t))$$

A constant step size turns the MC estimate into an exponentially weighted moving average of past returns

- Decays *geometrically* with visits - the return from k visits ago depends on $\alpha(1-\alpha)^k$), instead of *arithmetically* according to N returns having an equal weight of $\frac{1}{N}$

In general, we *prefer* non-stationary estimators because our policy we will be evaluating is continuously improving

- Essentially we are always in a non-stationary setting in RL as we improve our policy through experience

In summary, in Monte-Carlo learning we

1. Run out episodes,
2. look at the complete returns, and
3. update estimates of the mean value of return at each state of the return.

Temporal-Difference Learning

Temporal-Difference (TD) Learning

TD methods learn directly from episodes of experience

- TD is *model-free*: no knowledge of transitions/rewards (as in MC)

TD learns from *incomplete* episodes, by bootstrapping

- It substitutes, or bootstraps, reminder of the trajectory with the *estimate* of what will happen, instead of waiting for full returns
- i.e. TD updates one guess with a subsequent guess

MC versus TD

Goal: learn v_π online from experience under policy π

Incremental every-visit Monte-Carlo:

Update value $V(S_t)$ toward *actual* return G_t

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t - V(S_t) \right)$$

Simplest temporal-difference learning algorithm: TD(0):

Update value $V(S_t)$ toward *estimated* return $R_{t+1} + \gamma V(S_{t+1})$ (*like Bellman equation*)

$$V(S_t) \leftarrow V(S_t) + \alpha \left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \right)$$

$R_{t+1} + \gamma V(S_{t+1})$ is called the TD *target* (*we are moving towards*)
$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$ is called the TD *error*

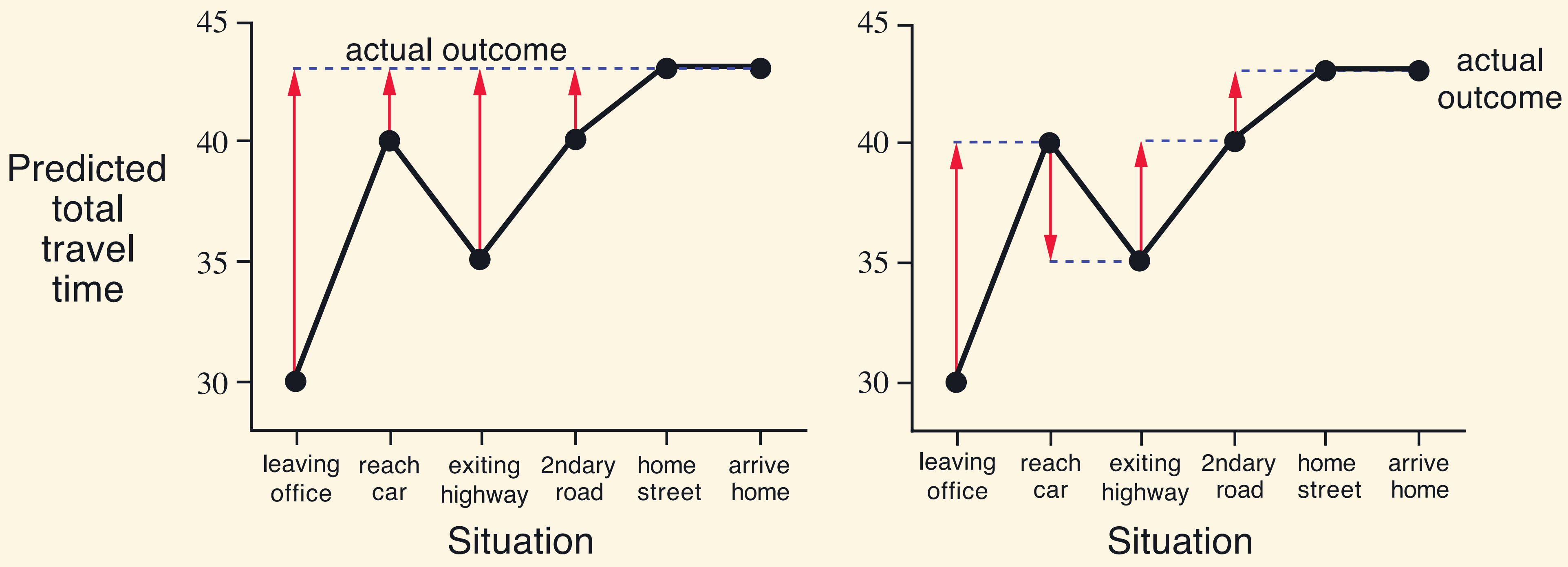
Driving Home Example

State	Elapsed Time (min)	Predicted Time to Go	Predicted Total Time
leaving office	0	30	30
reach car, raining	5	35	40
exit highway	20	15	35
behind truck	30	10	40
home street	40	3	43
arrive home	43	0	43

Driving Home Example: MC versus TD

Changes recommended by Monte Carlo methods ($\alpha = 1$):

Changes recommended by TD methods ($\alpha = 1$):



Red arrow represent recommended updates by MC and TD respectively

Advantages & Disadvantages of MC versus TD

TD can learn *before* knowing the final outcome

- TD can learn online after every step
- MC must wait until end of episode before return is known

TD can learn *without* the final outcome

- TD can learn from incomplete sequences
- MC can only learn from complete sequences
- TD works in continuing (non-terminating) environments
- MC only works for episodic (terminating) environments

Bias/Variance Trade-Off

Return $G_t = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{T-1} R_T$ is an *unbiased* estimate of $v_\pi(S_t)$.

True TD target $R_{t+1} + \gamma\, v_\pi(S_{t+1})$ is an *unbiased* estimate of $v_\pi(S_t)$.

- TD target $R_{t+1} + \gamma\, V(S_{t+1})$ is a *biased* estimate of $v_\pi(S_t)$.
- TD target has much lower variance than the return, since
  - Return depends on *many* random actions, transitions, rewards.
  - TD target depends on *one* random action, transition, reward.

Advantages & Disadvantages of MC versus TD (2)

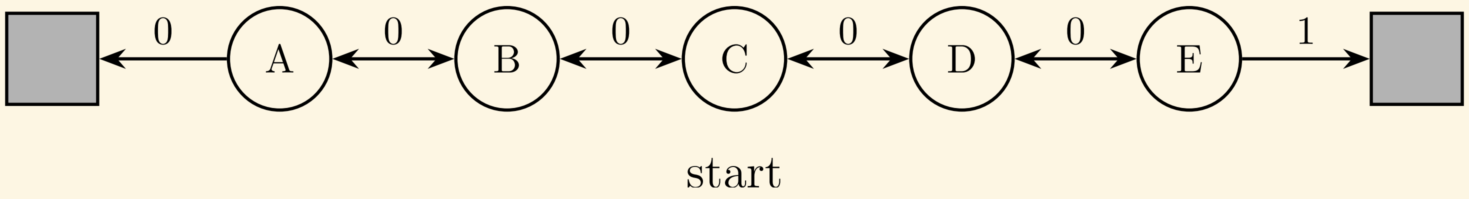
MC has high variance, zero bias

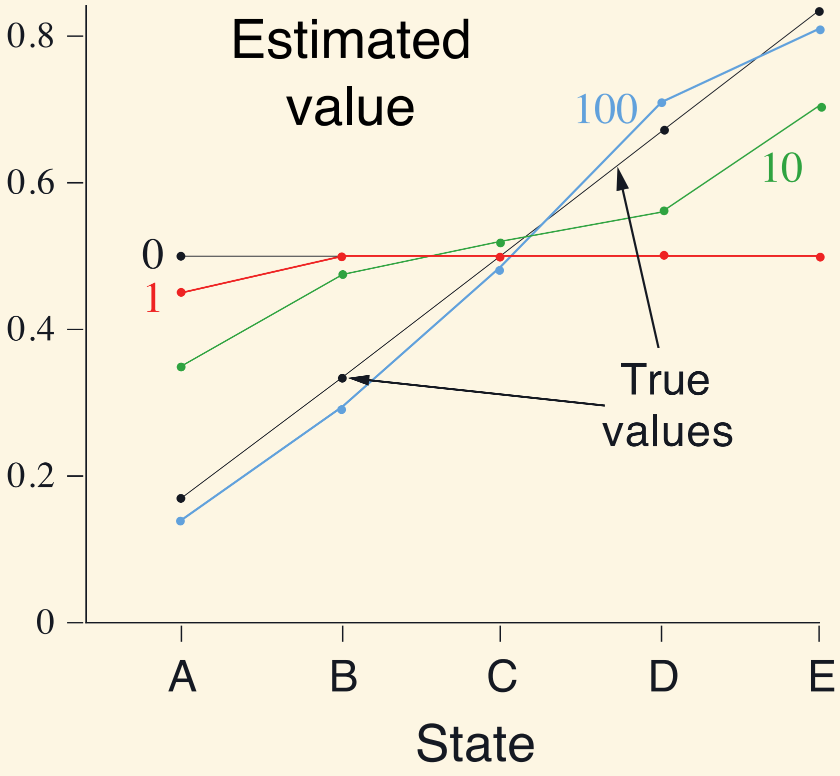
- Good convergence properties (even with function approximation)
- Not very sensitive to initial value
- Very simple to understand and use

TD has low variance, *some* bias

- Usually more efficient than MC
- TD(0) converges to $v_\pi(s)$ (but not always with function approximation)
- More sensitive to initial value

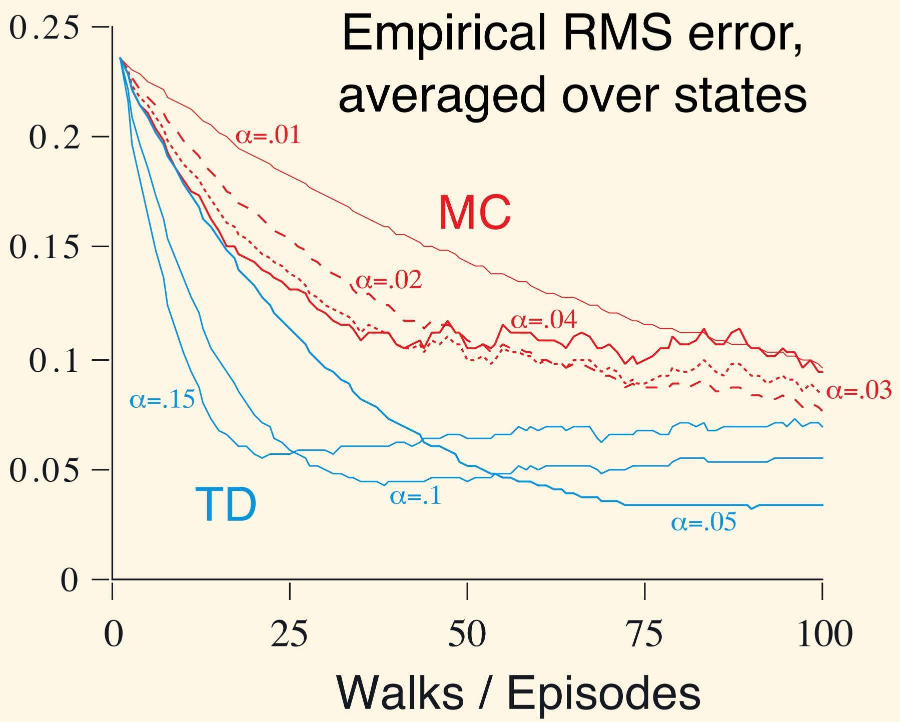
Random Walk Example





Random Walk example based on uniform random policy: left 0.5 and right 0.5

Random Walk: MC versus TD



- This demonstrates the benefit of bootstrapping

Batch MC and TD

MC and TD both converge in the limit

- $V(s) \rightarrow v_\pi(s)$ as experience $\rightarrow \infty$

What about a *batch* solution for finite experience, k finite episodes?

$$s_1^1, a_1^1, r_2^1, \cdots s_{T_1}^1$$

$$\vdots$$

$$s_1^K, a_1^K, r_2^K, \cdots s_{T_K}^K$$

In *batch* mode you repeatedly sample episode $k \in [1, K]$ and apply MC or TD(0) to episode k

AB Example

Two states A, B; no discounting; 8 episodes:

$A, 0, B, 0$

$B, 1$

$B, 1$

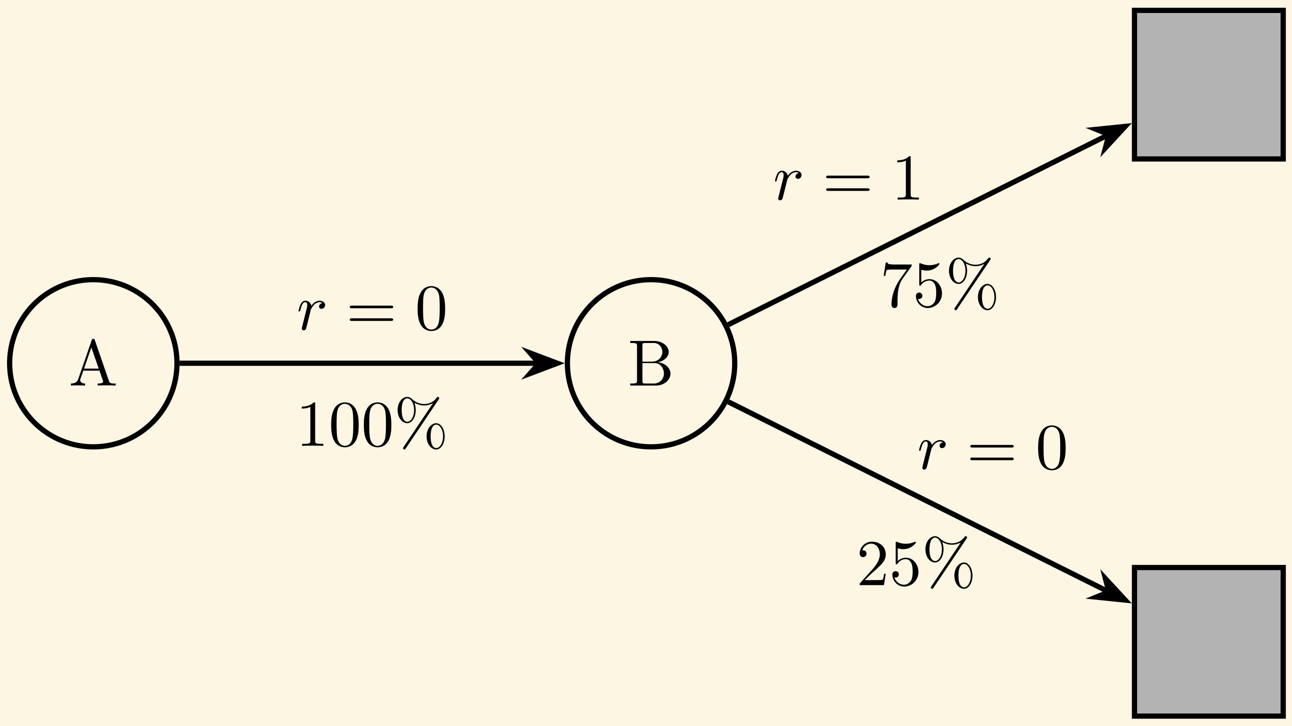
$B, 1$

$B, 1$

$B, 1$

$B, 1$

$B, 0$



Clearly, $V(B) = \frac{6}{8} = 0.75$, but what about $V(A)$?

Certainty Equivalence

MC converges to solution with minimum mean-squared error

- Best fit to the observed returns

$$\sum_{k=1}^{K} \sum_{t=1}^{T_k} \left(G_t^k - V(s_t^k)\right)^2$$

- In the AB example, $V(A) = 0$

TD(0) converges to solution of max likelihood Markov model that best explains the data

- Solution to the MDP $\langle \mathcal{S}, \mathcal{A}, \hat{\mathcal{P}}, \hat{\mathcal{R}}, \gamma \rangle$ that best fits the data ($\hat{\mathcal{P}}$ counts the transitions, and $\hat{\mathcal{R}}$ the rewards)

$$\hat{\mathcal{P}}^a_{s,s'} = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{T_k} \mathbf{1}(s^k_t, a^k_t, s^k_{t+1} = s, a, s')$$

$$\hat{\mathcal{R}}^a_s = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{T_k} \mathbf{1}(s^k_t, a^k_t = s, a) r^k_t$$

- In the AB example, $V(A) = 0.75$

Advantages and Disadvantages of MC versus TD (3)

TD exploits Markov property

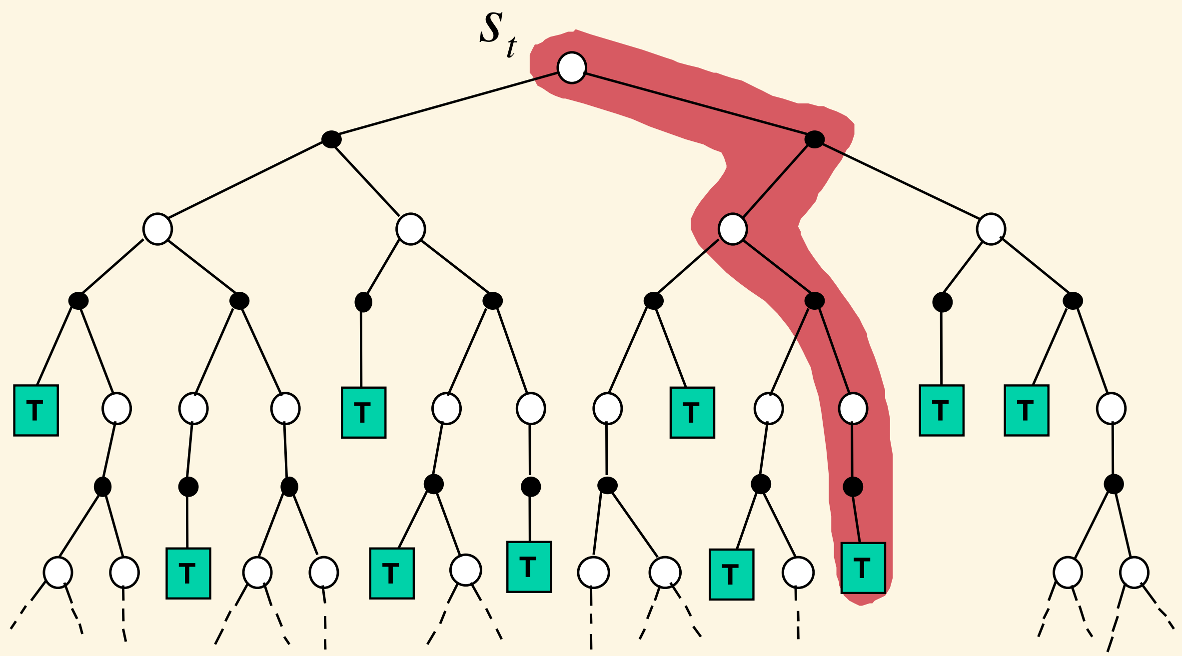
- Usually more efficient in Markov environments

MC does not exploit Markov property

- Usually more effective in non-Markov environments
- Note that *partial observability* and *non-stationarity* are reasons an environment can be non-Markov

Monte-Carlo Backup

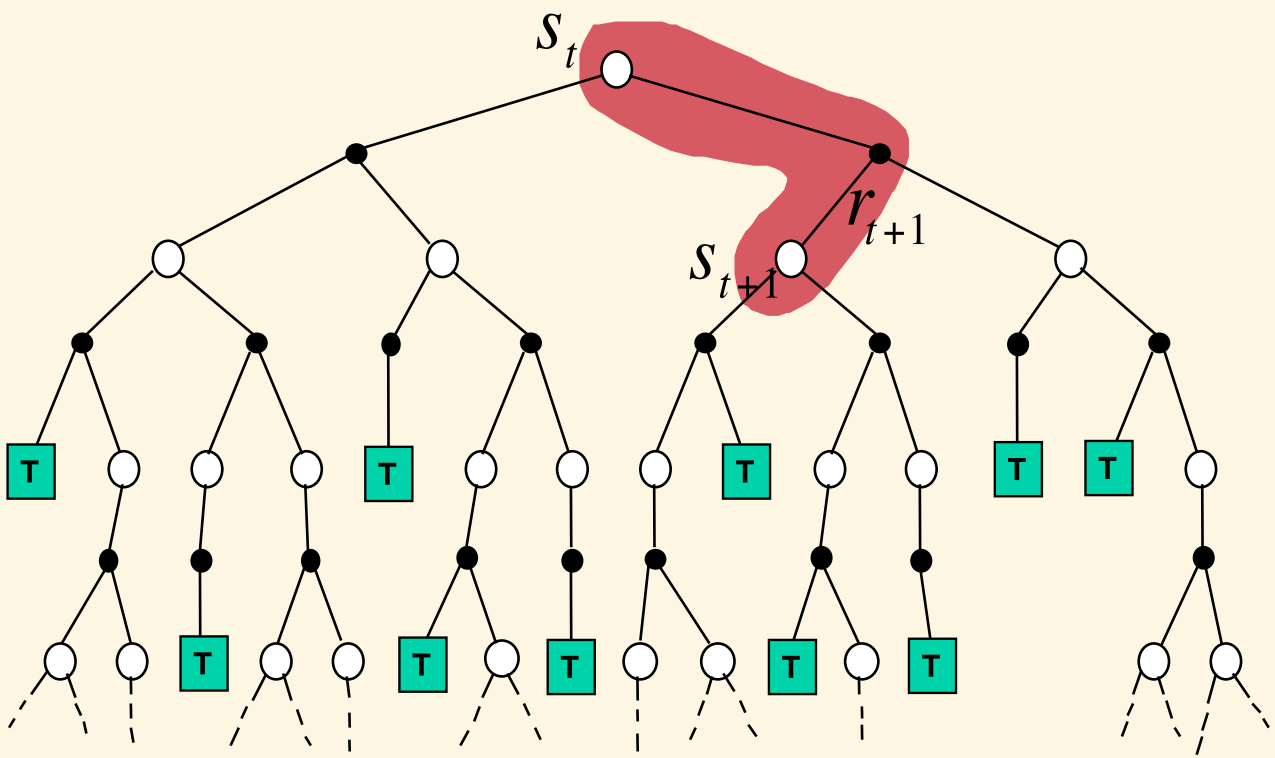
$$V(S_t) \leftarrow V(S_t) + \alpha\big(G_t - V(S_t)\big)$$



Starting at one state, sample one complete trajectory to update the value function

Temporal-Difference Backup

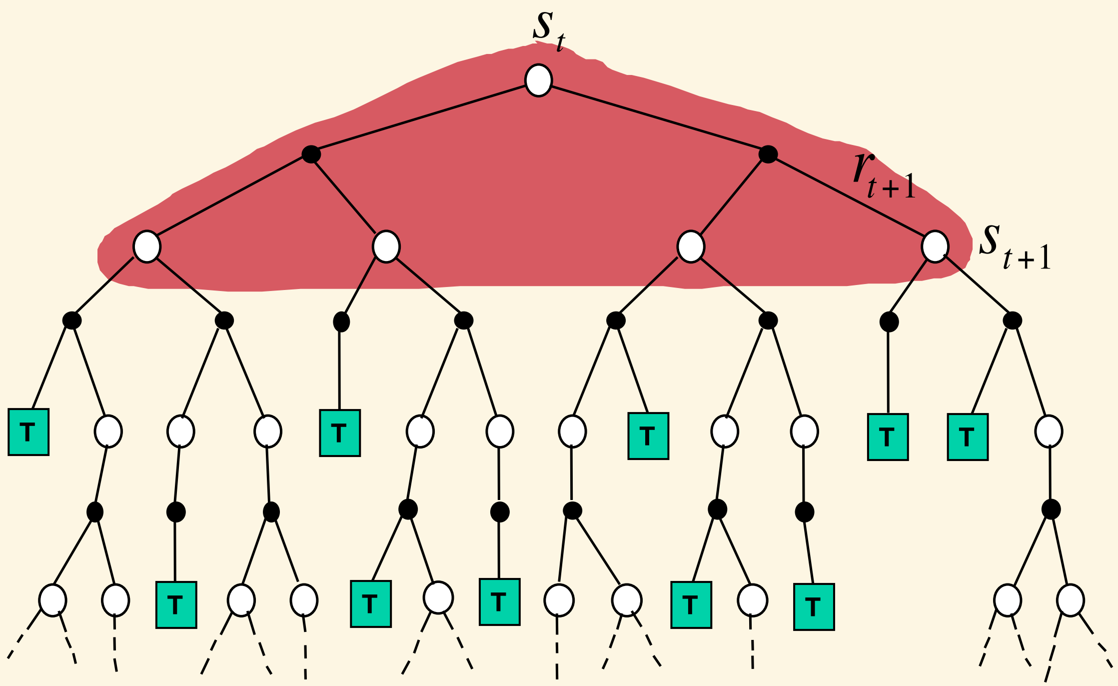
$$V(S_t) \leftarrow V(S_t) + \alpha\left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right)$$



In TD backup is just over one step

Tree Search/Dynamic Programming Backup

$$V(S_t) \leftarrow \mathbb{E}_\pi[R_{t+1} + \gamma V(S_{t+1})]$$



If we know the dynamics of the environment, we can do search and a complete backup over the tree

Bootstrapping and Sampling

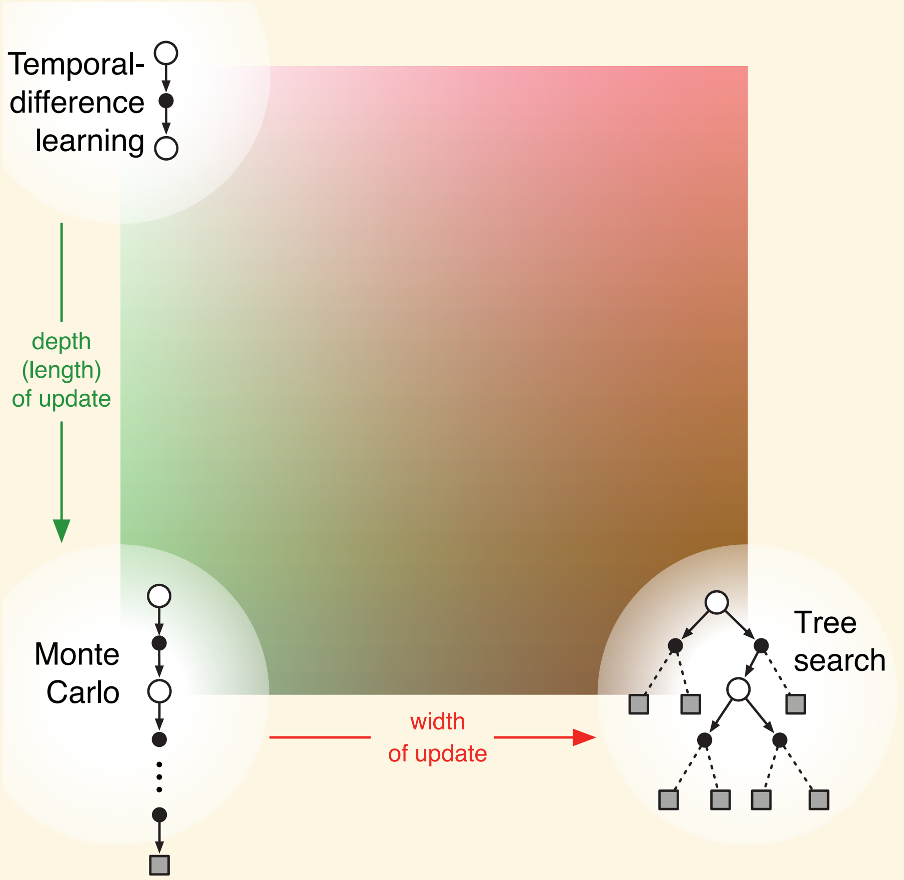
Bootstrapping: update involves an *estimate*

- MC does not bootstrap
- Tree Search (with heuristic search) or dynamic programming bootstraps
- TD bootstraps

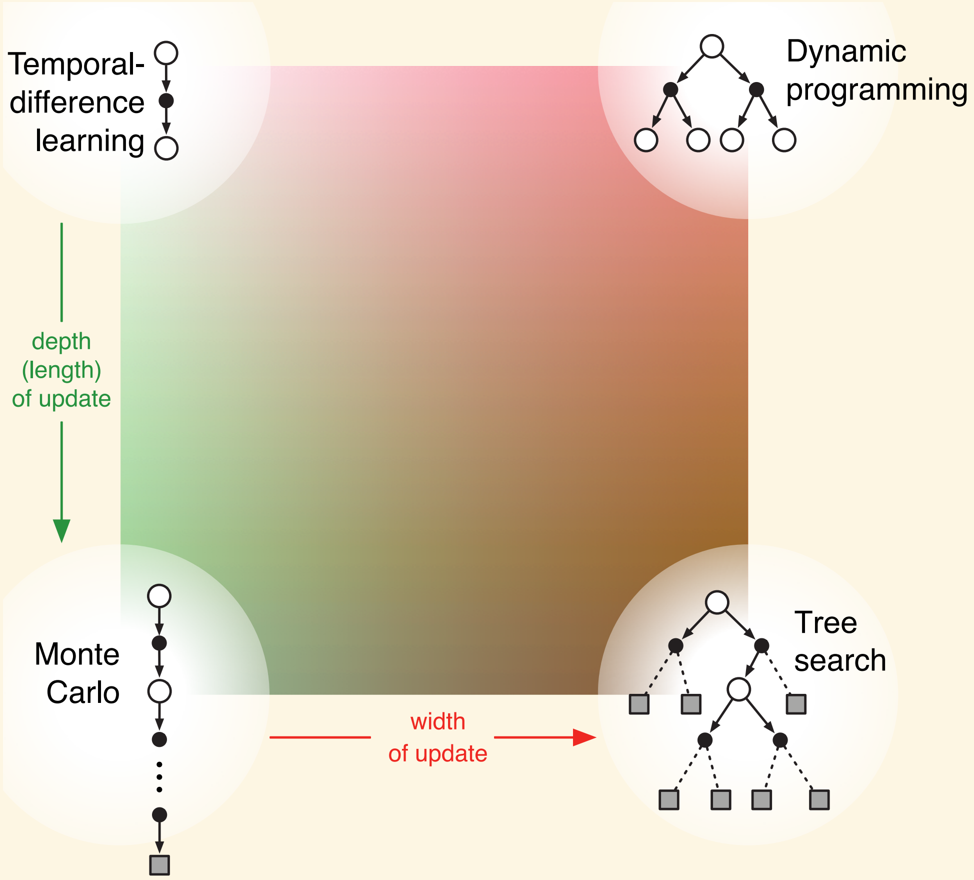
Sampling: update samples an *expectation*

- MC samples
- Tree Search does not sample
- TD samples

Unified View of Reinforcement Learning (1)



Unified View of Reinforcement Learning (2)

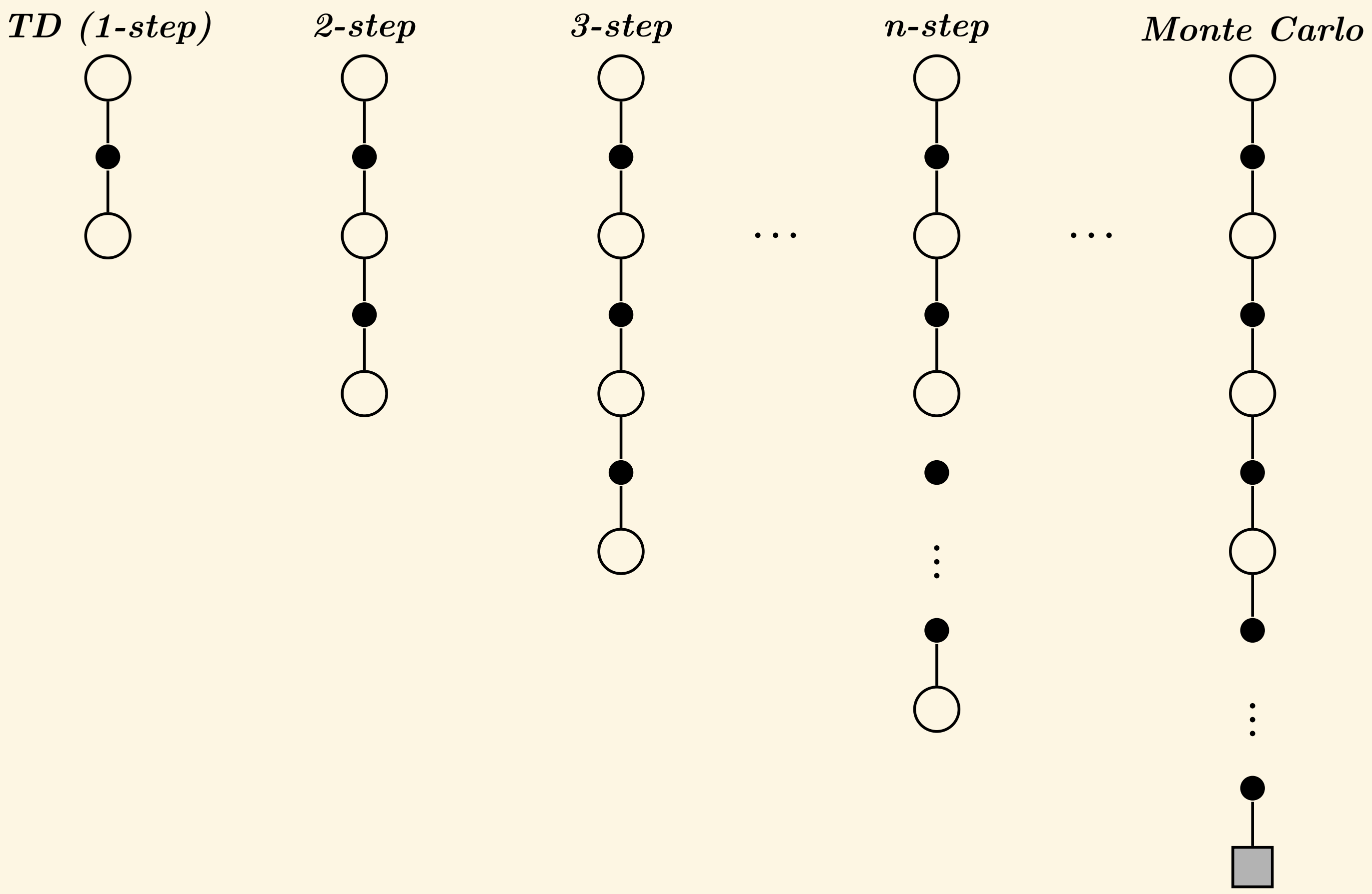


- Dynamic programming only explores one level, in it’s most simple form.
- In practice, dynamic programming is used during tree search, similar to classical planning.

TD(λ)

n-Step Prediction

Let TD target look n steps into the future



n-Step Return

Consider the following n-step returns for $n = 1, 2, \infty$:

$$
\begin{aligned}
&n = 1 \text{ (TD(0))} && G_t^{(1)} = R_{t+1} + \gamma V(S_{t+1}) \\
&n = 2 && G_t^{(2)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2}) \\
&\vdots && \vdots \\
&n = \infty \text{ (MC)} && G_t^{(\infty)} = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{T-1} R_T
\end{aligned}
$$

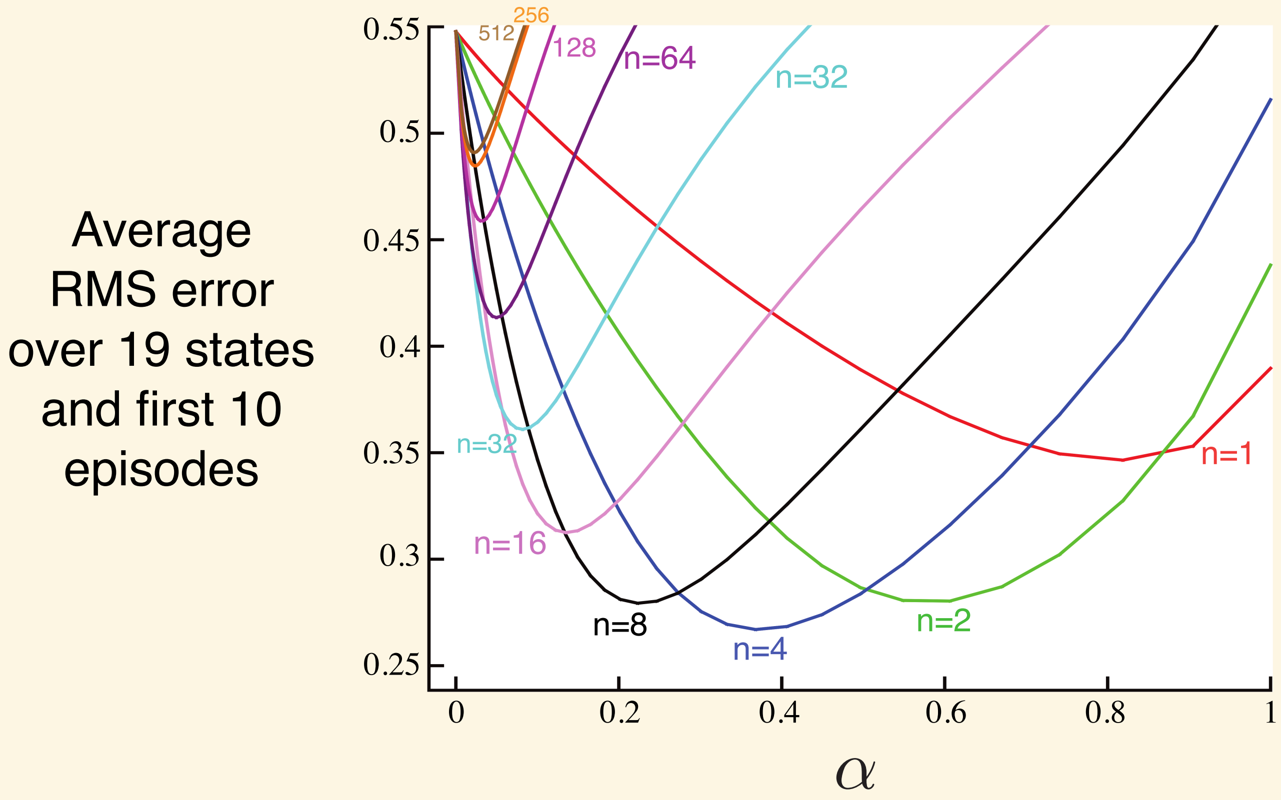
Define the n-step Return (real reward + estimated reward, $\gamma^n V(S_{t+n})$):

$$G_t^{(n)} = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

n-step temporal-difference learning (updated in direction of error)

- MC looks at the real reward
- So which n is the best?

Large Random Walk Example



- RMS errors vary according to step size α, with the *optimum* dependent on n

Note that RMS errors also vary according to whether learning is *on-line* or *off-line* updates (not shown here)

- i.e. whether *immediately* update value function or *defer* updates until episode ends

Averaging n-Step Returns

We can form *mixtures* of different n:

e.g. average of 2-step and 4-step returns:

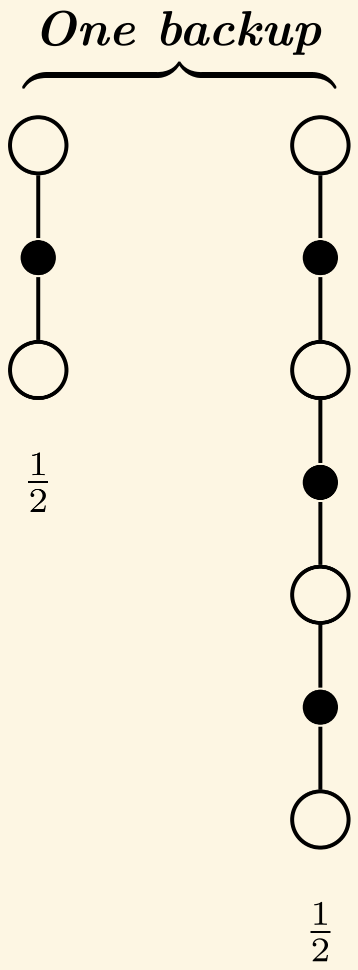
$$\frac{1}{2}G_t^{(2)} + \frac{1}{2}G_t^{(4)}$$

We can average n-step returns over different n

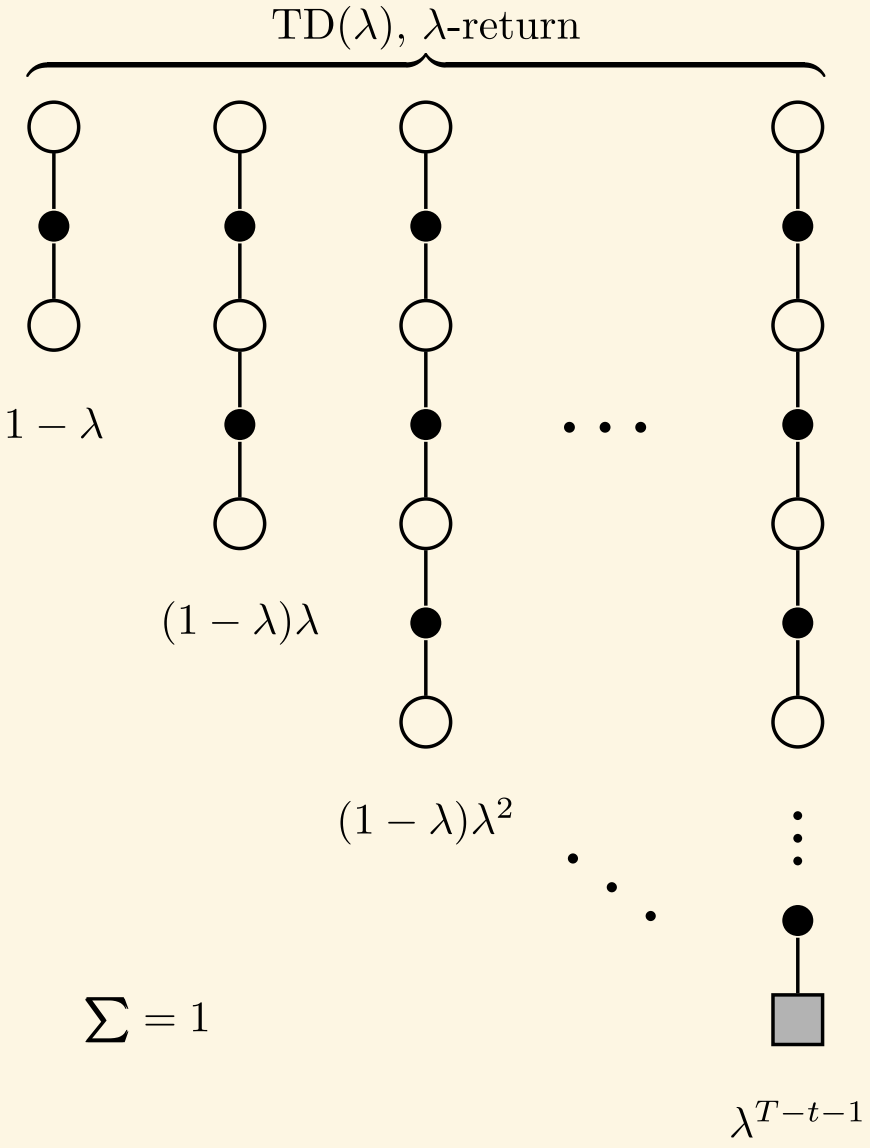
- e.g. average the 2-step and 4-step returns

Combines information from two different time-steps

Can we efficiently combine information from *all* time-steps to be more robust?



λ-return



The λ-return G_t^λ combines *all* n-step returns $G_t^{(n)}$

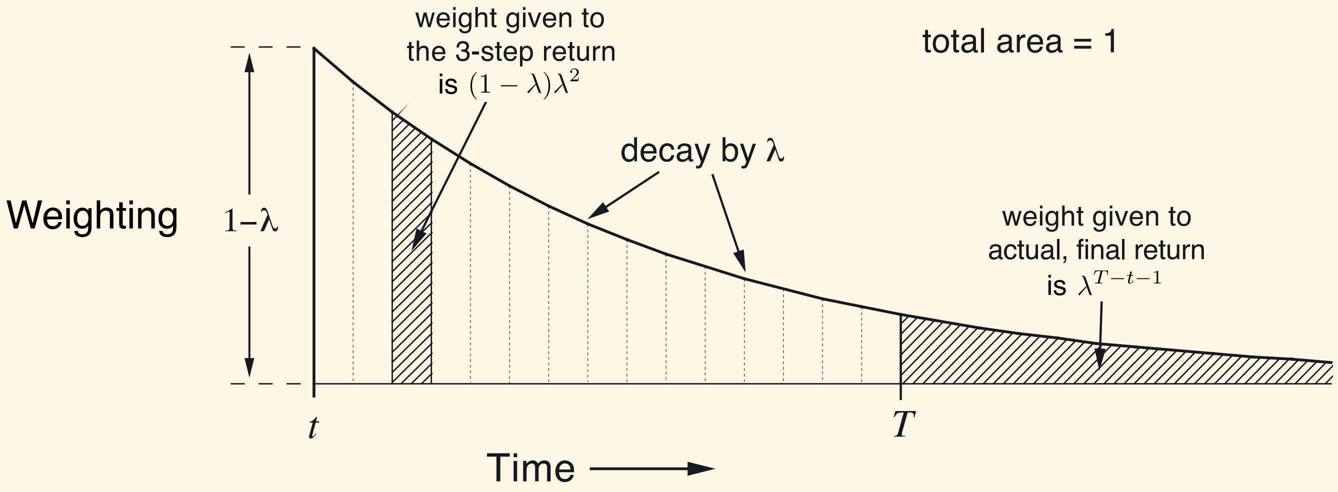
- Using *weight* $(1-\lambda)\lambda^{n-1}$

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}G_t^{(n)}$$

Forward-view TD(λ)

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t^\lambda - V(S_t)\right)$$

TD(λ) Weighting Function



$$G_t^{\lambda} = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)}$$

- λ-return is a geometrically weighted return for every n-step return

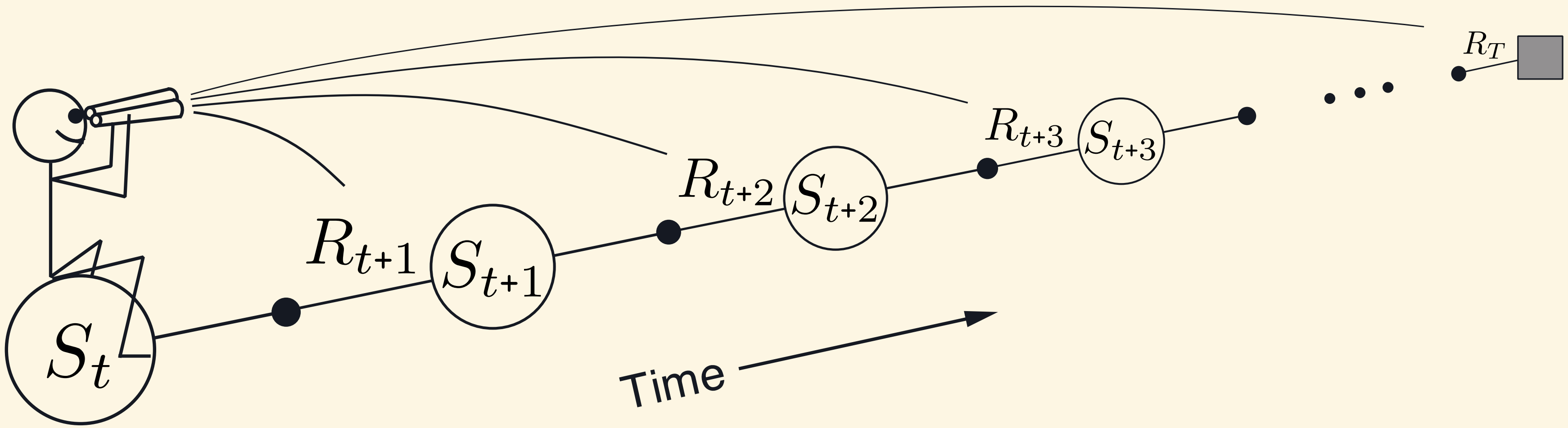
Note that geometric weightings are *memoryless*

- i.e. we can compute $TD(\lambda)$ with with no greater complexity than $TD(0)$

However, the formulation presented so far is a *forward view*

- we can't look into the future!

Forward View of TD(λ)

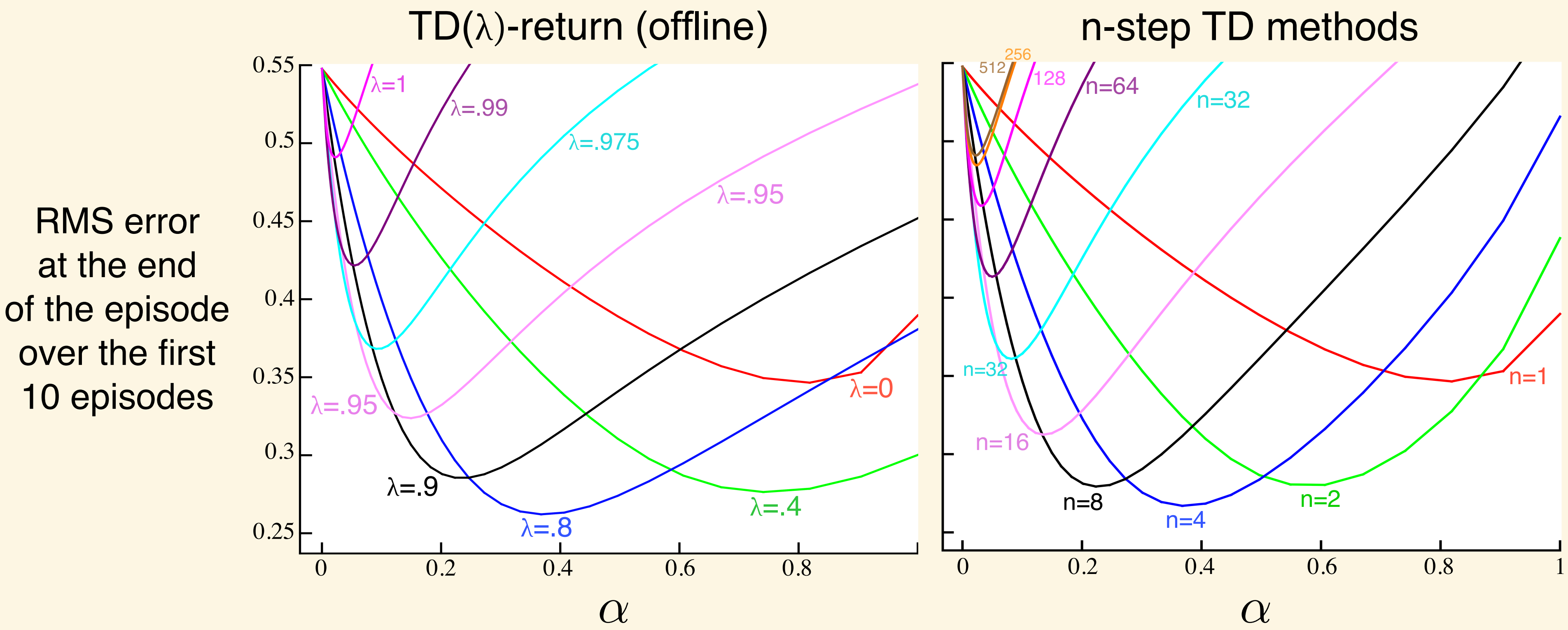


Update value function towards the λ-return

- Forward-view looks into the future to compute G_t^λ
- Like MC, can only be computed from complete episodes

We will see shortly how an *iterative* algorithm achieves the forward view without having to wait until the future

Forward View of TD(λ) on Large Random Walk



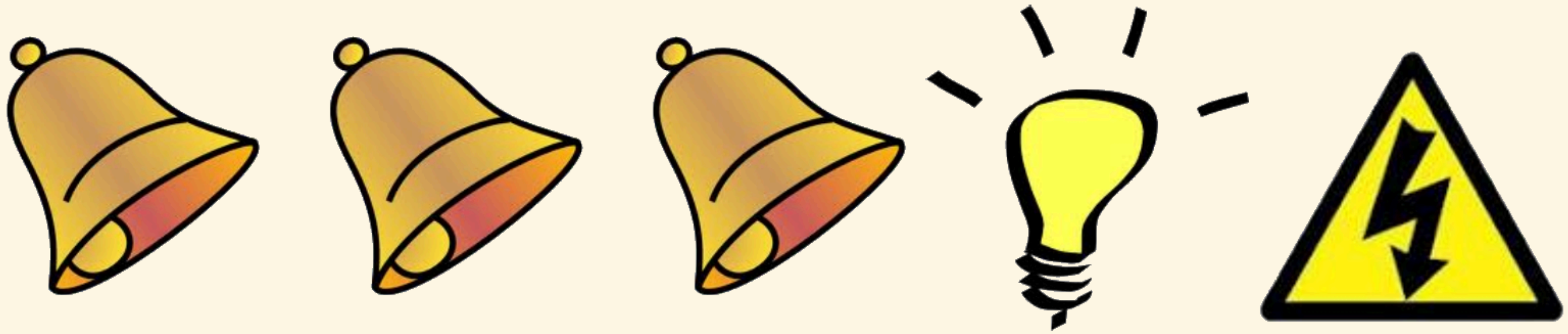
We can see using TD(λ), and choosing λ value (left hand side), is more robust than choosing a *unique* n-step value (right hand side)

- $\lambda = 1$ is MC and $\lambda = 0$ is TD(0)

Backward View of TD(λ)

- Forward view provides theory
- Backward view provides mechanism
- Update online, every step, from incomplete sequences

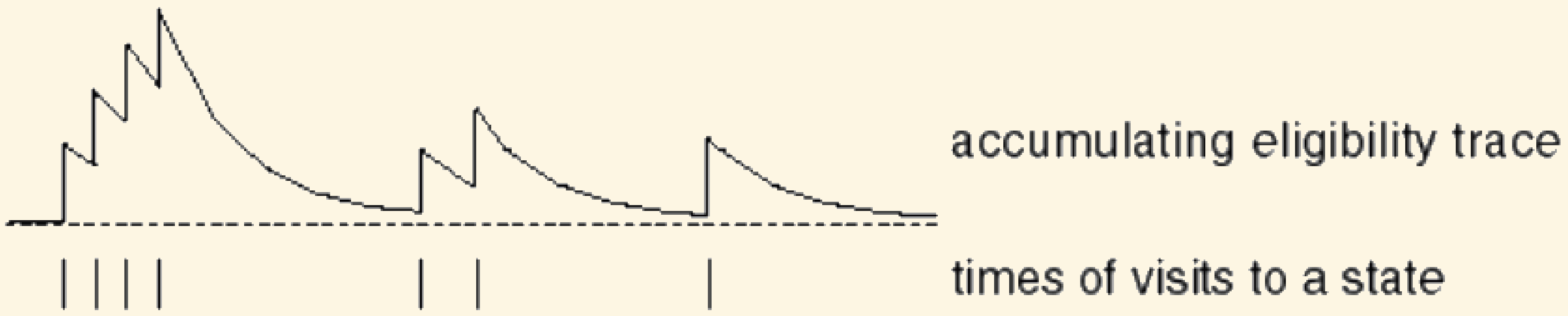
Eligibility Traces



Credit assignment problem: *did bell or light cause shock?*

- Frequency heuristic: assign credit to most frequent states
- Regency heuristic: assign credit to most recent states
- Eligibility traces *combine* both heuristics

$$E_0(s) = 0$$
$$E_t(s) = \gamma\lambda E_{t-1}(s) + 1(S_t = s)$$

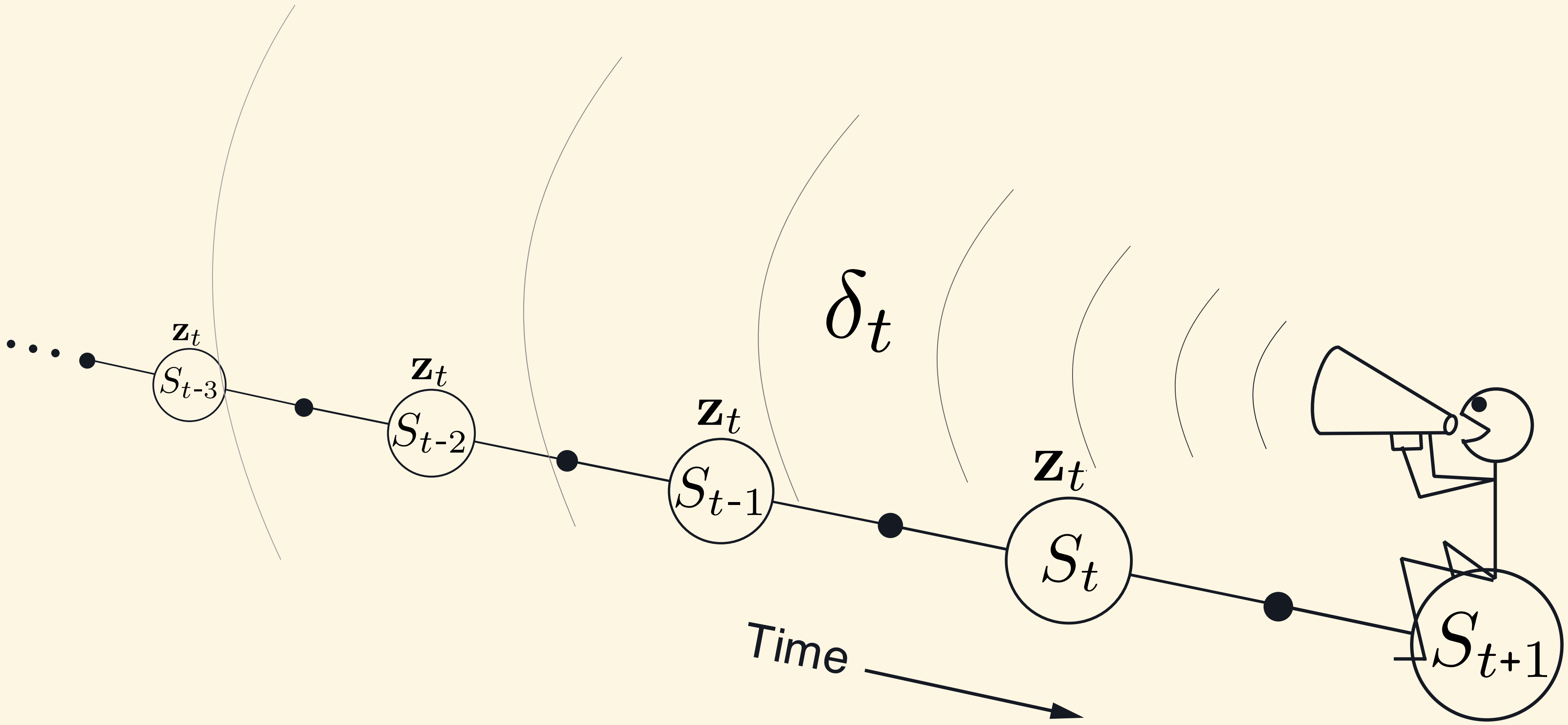


Backward View TD(λ)

- Keep an eligibility trace for every state s
- Update value $V(s)$ for every state s
- In proportion to TD-error δ_t and eligibility trace $E_t(s)$

$$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$$

$$V(s) \leftarrow V(s) + \alpha\,\delta_t E_t(s)$$



$\mathbf{z}_t$
$S_{t\text{-}3}$
$\mathbf{z}_t$
$S_{t\text{-}2}$
$\mathbf{z}_t$
$S_{t\text{-}1}$
δ_t
$\mathbf{z}_t$
S_t
S_{t+1}
Time

TD(λ) and TD(0)

When $\lambda = 0$, only the current state is updated.

$$E_t(s) = \mathbf{1}(S_t = s)$$

$$V(s) \leftarrow V(s) + \alpha\,\delta_t\,E_t(s)$$

This is exactly equivalent to the TD(0) update.

$$V(S_t) \leftarrow V(S_t) + \alpha\,\delta_t$$

TD(λ) and MC

When $\lambda = 1$, credit is deferred until the end of the episode.

- Consider episodic environments with offline updates.
- Over the course of an episode, the total update for TD(λ) is the same as the total update for MC.

Theorem

The sum of offline updates is identical for forward-view and backward-view TD(λ):

$$\sum_{t=1}^{T} \alpha\, \delta_t\, E_t(s) \;=\; \sum_{t=1}^{T} \alpha\big(G_t^{\lambda} - V(S_t)\big)\, \mathbf{1}(S_t = s).$$

Example: Temporal-Difference Search for MCTS

Example: Temporal-Difference Search for MCTS

Simulation-based search

. . . using TD instead of MC (bootstrapping)

- MC tree search applies MC control to sub-MDP from now
- TD search applies Sarsa to sub-MDP from now

MC versus TD search

For model-free reinforcement learning, bootstrapping is helpful

- TD learning reduces variance but increases bias
- TD learning is usually more efficient than MC
- TD(λ) can be much more efficient than MC

For simulation-based search, bootstrapping is also helpful

- TD search reduces variance but increases bias
- TD search is usually more efficient than MC search
- TD(λ) search can be much more efficient than MC search

TD search

Simulate episodes from the current (real) state s_t

Estimate action-value function $Q(s, a)$

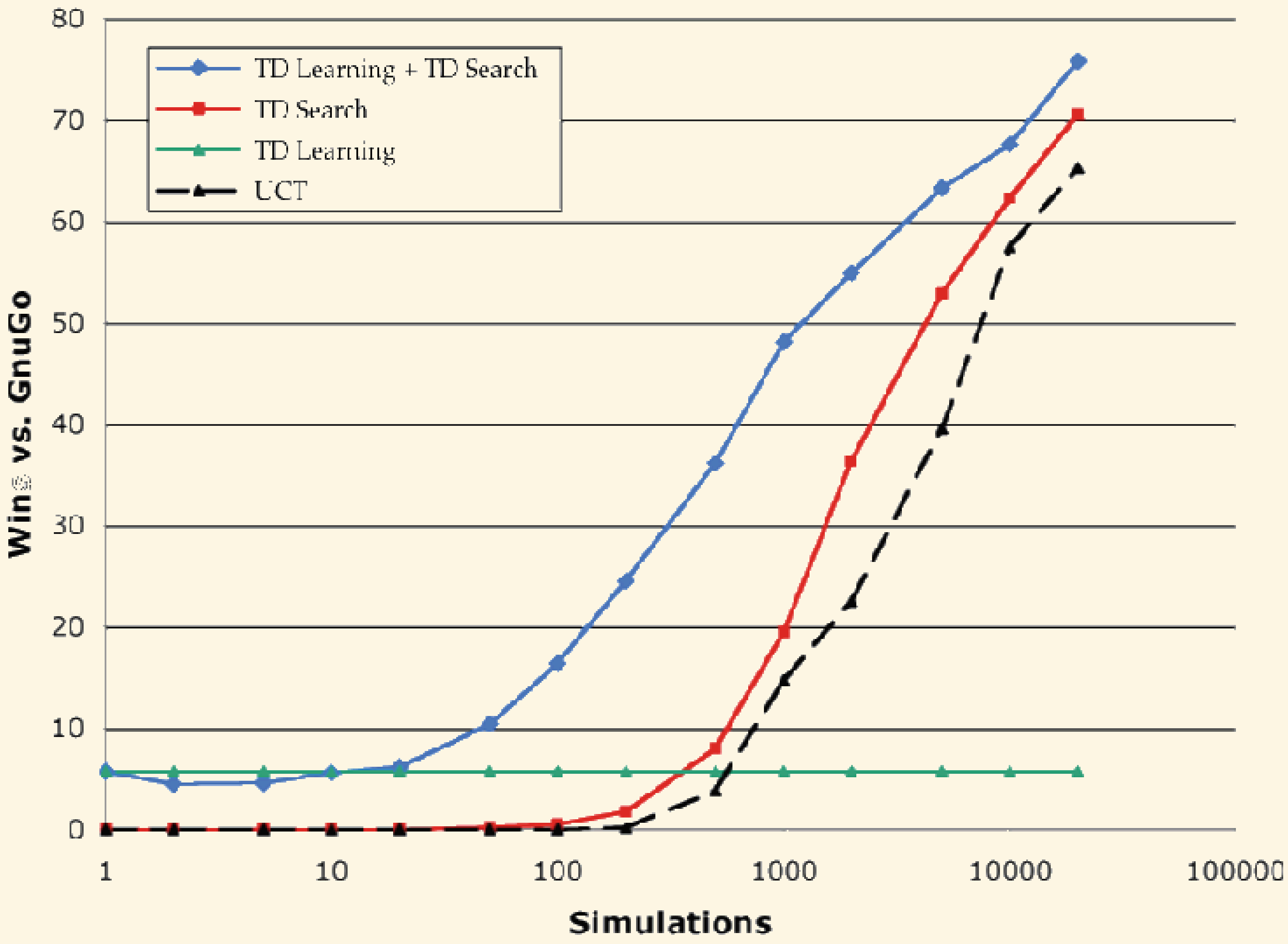
- For each step of simulation, update action-values by Sarsa

$$\Delta Q(S, A) = \alpha\big(R + \gamma Q(S', A') - Q(S, A)\big)$$

- Select actions based on action-values $Q(s, a)$
  - e.g. ϵ-greedy

May also use function approximation for Q, if needed

Results of TD search in Go



- Black dashed line is MCTS
- Blu line is Dyna-Q (not covered in this module)

Learning from simulation is an effective method in search

Model-Free Control (Sarsa and Q-Learning)

Uses of Model-Free Control

Example problems that can be naturally modelled as MDPs

- Elevator
- Parallel Parking
- Ship Steering
- Bioreactors
- Power stations
- Computer Programming
- Fine tuning LLMs
- Portfolio management
- Protein Folding
- Robot walking

For most of these problems, either:

- MDP model is unknown, but experience can be sampled
- MDP model is known, but is too big to use, except by samples

Model-free control can solve these problems

On and Off-Policy Learning

On-policy learning

- "Learn on the job"
- Learn about policy π from experience sampled from π

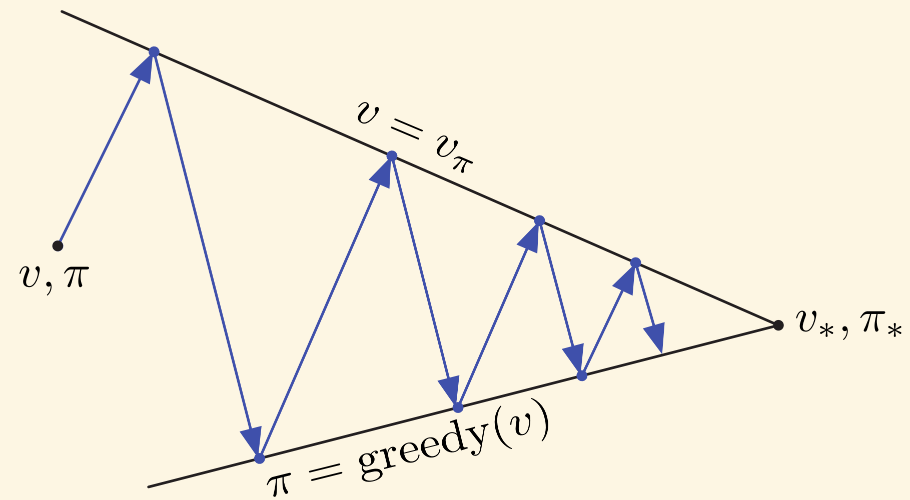
Off-policy learning

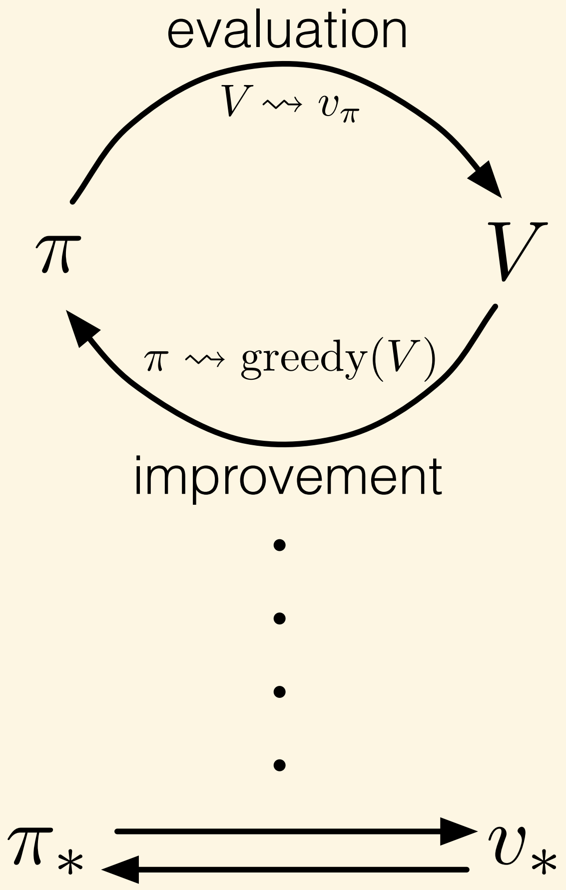
- "Look over someone's shoulder"
- Learn about policy π from experience sampled from μ

Off-policy learning uses trajectories sampled from policy μ, e.g. from another robot, AI agent, human, or simulator.

On-Policy Monte-Carlo Control

Generalised Policy Iteration





Alternation *converges* on optimal policy π_*

- Policy evaluation Estimate v_π
  e.g. Iterative policy evaluation, *going up*
- Policy improvement Generate $\pi' \geq \pi$
  e.g. Greedy policy improvement, act greedily with respect to value function, *going down*

Principle of Optimality

Any optimal policy can be subdivided into two components:

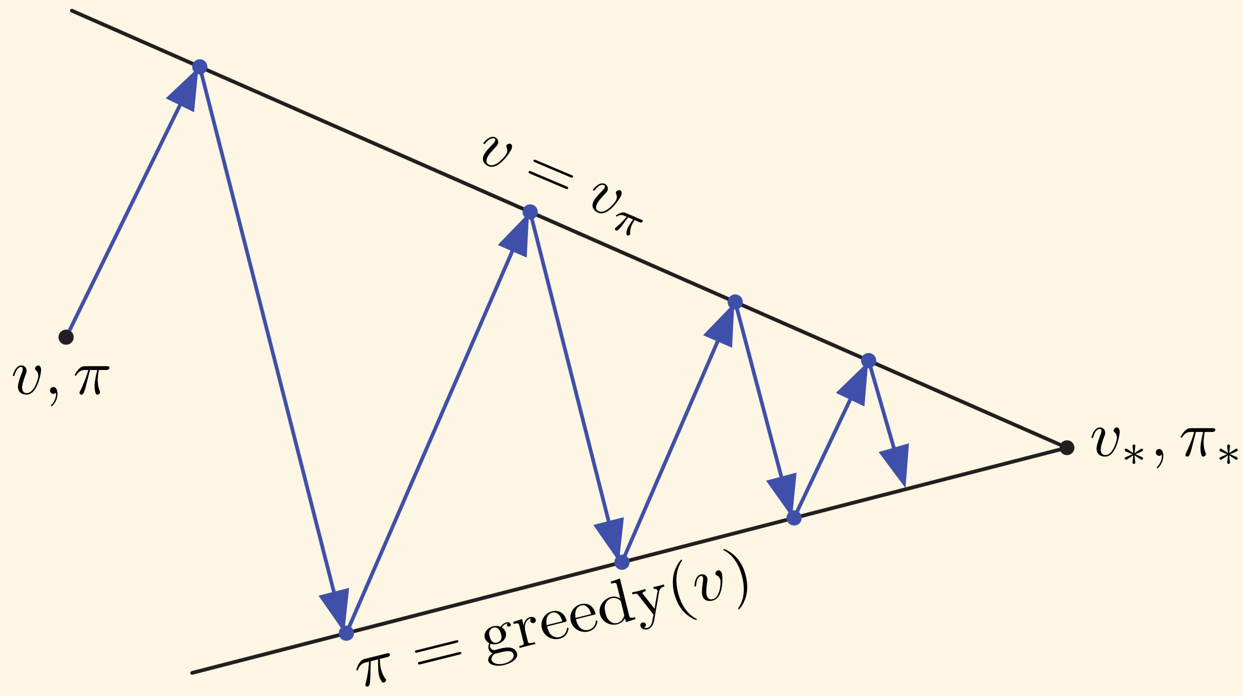
- An optimal first action A_*
- Followed by an optimal policy from successor state S'

Theorem (Principle of Optimality)

A policy $\pi(a|s)$ achieves the optimal value from state s, $v_\pi(s) = v_*(s)$, if and only if

- For any state s' reachable from s
- π achieves the optimal value from state s', $v_\pi(s') = v_*(s')$

Generalised Policy Iteration with Monte-Carlo Evaluation



Policy evaluation 1. Can we use Monte-Carlo policy evaluation to estimate $V = v_\pi$(running multiple episodes/rollouts)?

Policy improvement 2. Can we do greedy policy improvement with MC evaluation?

Model-Free Policy Iteration Using Action-Value Function

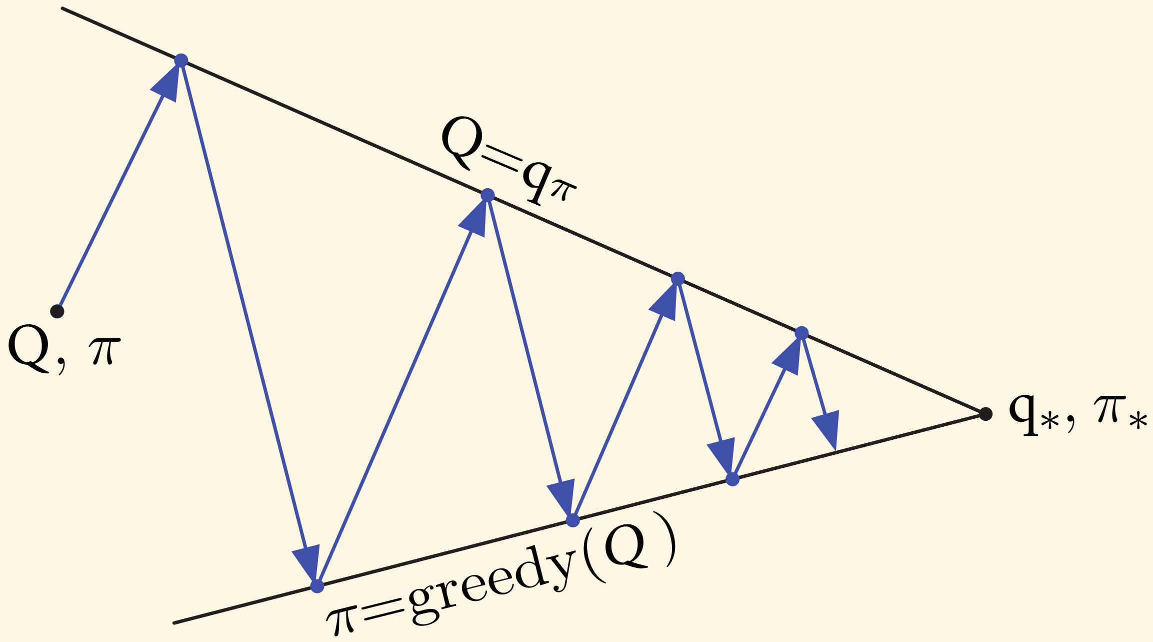
- Problem 1: Greedy policy improvement over $V(s)$ requires a model of MDP

$$\pi'(s) = \underset{a \in \mathcal{A}}{\arg\max} \left[\mathcal{R}_s^a + \sum_{s'} P_{ss'}^a V(s') \right]$$

- Alternative: use action-value functions in place of model
  Greedy policy improvement over $Q(s,a)$ is *model-free*

$$\pi'(s) = \underset{a \in \mathcal{A}}{\arg\max}\, Q(s,a)$$

Generalised Policy Iteration with Action-Value Function



Policy evaluation We run Monte-Carlo policy evaluation using $Q = q\pi$

- For each state-action pair $Q(A, S)$ we take *mean* return
- We do this for *all* states and actions, i.e. we don't need model

Policy improvement Greedy policy improvement?

Problem 2: We are acting *greedily* which means you can get stuck in *local minima*

- Note that at each step we are running *episodes* for the policy by trial and error, so we might not see some states
- i.e. you won't necessarily see the states you need in order to get get correct estimate of value function
- Unlike in dynamic programming where you see all states

Example of Greedy Action Selection (Bandit problem)

"Behind one door is tenure - behind the other is flipping burgers at McDonald's."

There are two doors in front of you.

- You open the left door and get reward 0
  $V(left) = 0 \ (Monte\ Carlo\ Estimate)$
- You open the right door and get reward $+1$
  $V(right) = +1$
- You open the right door and get reward $+3$
  $V(right) = +2$
- You open the right door and get reward $+2$
  $V(right) = +2$

$\vdots$

You may never explore left door again!

- i.e. are you sure you've chosen the best door?

ε-Greedy Exploration

The simplest idea for ensuring continual exploration:

All m actions are tried with non-zero probability

- with probability $1 - \varepsilon$ choose the best action, *greedily*
- with probability ε choose a *random*

$$\pi(a \mid s) = \begin{cases} \dfrac{\varepsilon}{m} + 1 - \varepsilon & \text{if } a^* = \arg\max\limits_{a \in \mathcal{A}} Q(s, a) \\ \dfrac{\varepsilon}{m} & \text{otherwise} \end{cases}$$

ε-Greedy Policy Improvement

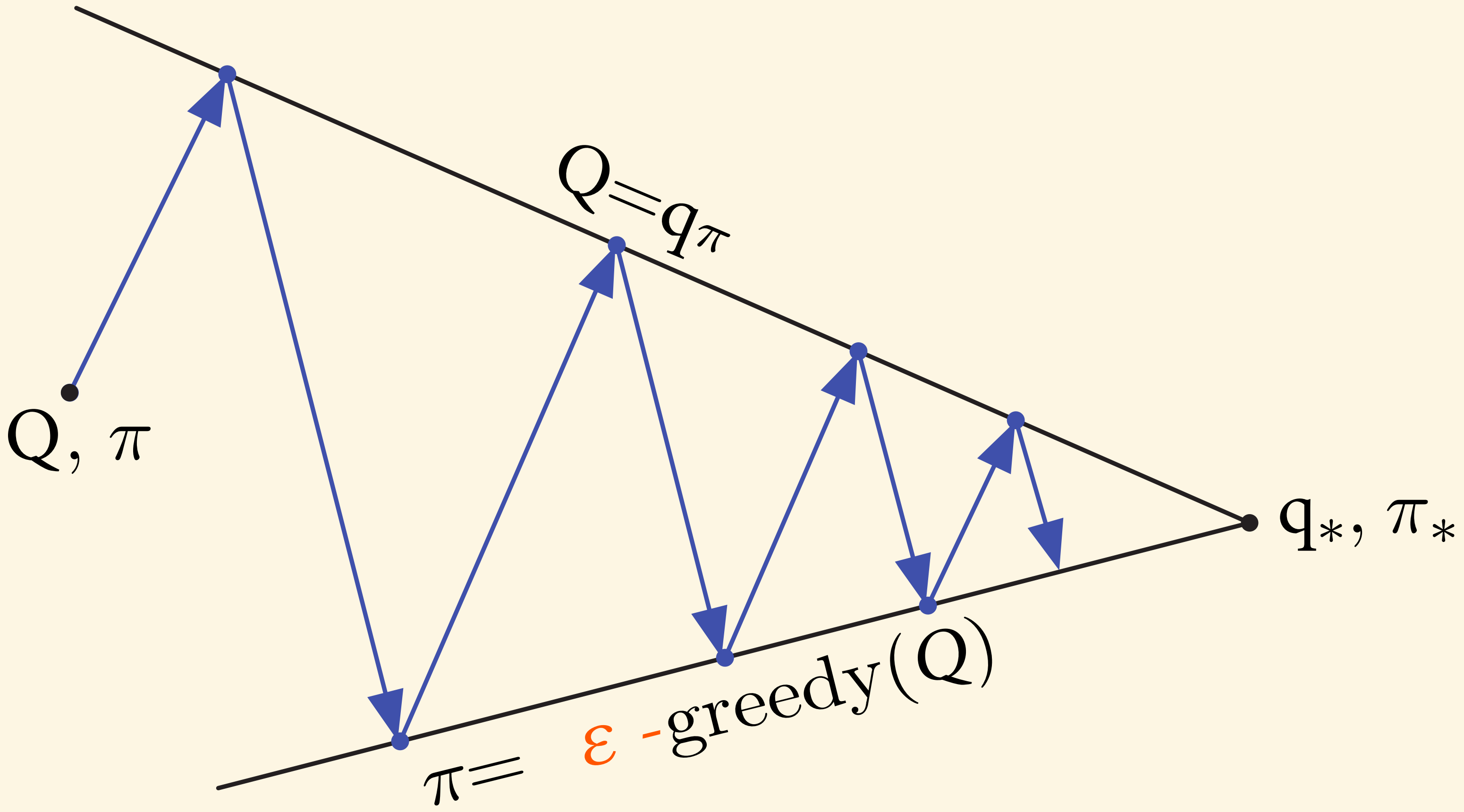
Theorem

For any ε-greedy policy π, the ε-greedy policy π' with respect to q_π is an improvement, $v_{\pi'}(s) \geq v_\pi(s)$

$$
\begin{aligned}
q_\pi(s, \pi'(s)) &= \sum_{a \in \mathcal{A}} \pi'(a \mid s)\, q_\pi(s, a) \\
&= \frac{\varepsilon}{m} \sum_{a \in \mathcal{A}} q_\pi(s, a) + (1 - \varepsilon) \max_{a \in \mathcal{A}} q_\pi(s, a) \\
&\geq \frac{\varepsilon}{m} \sum_{a \in \mathcal{A}} q_\pi(s, a) + (1 - \varepsilon) \sum_{a \in \mathcal{A}} \frac{\pi(a \mid s) - \frac{\varepsilon}{m}}{1 - \varepsilon} q_\pi(s, a) \\
&= \sum_{a \in \mathcal{A}} \pi(a \mid s)\, q_\pi(s, a) = v_\pi(s)
\end{aligned}
$$

Proof idea: $\max_{a \in \mathcal{A}} q_\pi(a, a)$ is at least as good as any weighted sum of all of your actions; therefore from the policy improvement theorem, $v_{\pi'}(s) \;\geq\; v_\pi(s)$

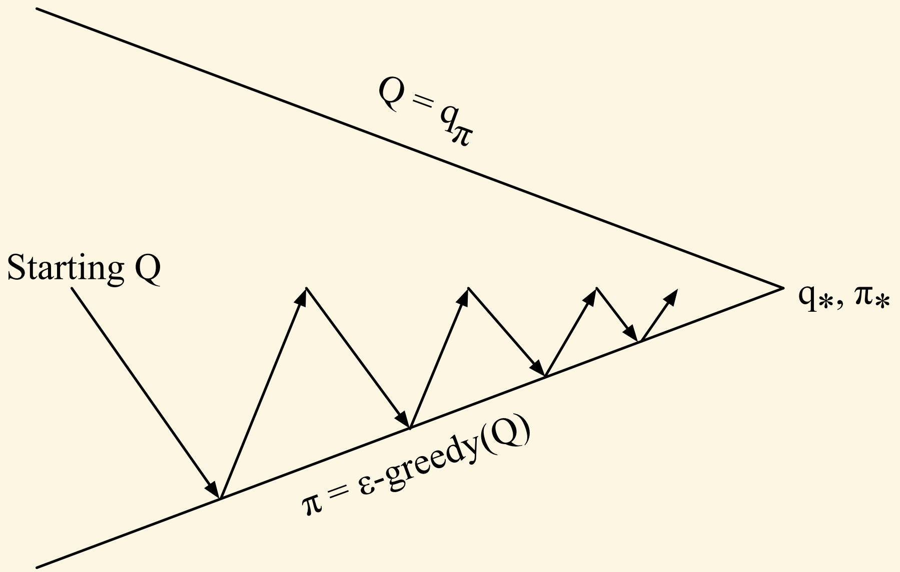
Monte-Carlo Policy Iteration



Policy evaluation Monte-Carlo policy evaluation, $Q = q_\pi$

Policy improvement $\varepsilon-$Greedy policy improvement

Monte-Carlo Control



Every episode:

Policy evaluation Monte-Carlo policy evaluation, $Q \approx q_\pi$

- Not necessary to fully evaluate policy every time, *going all the way to the top*, instead, immediately improve policy for every *episode*

Policy improvement $\varepsilon-$Greedy policy improvement

Greedy in the Limit with Infinite Exploration (GLIE)

Definition

Greedy in the Limit with Infinite Exploration (GLIE)

- All state–action pairs are explored infinitely many times,

$$\lim_{k \to \infty} N_k(s, a) = \infty$$

- The policy converges on a greedy policy,

$$\lim_{k \to \infty} \pi_k(a \mid s) = \mathbf{1}\left(a = \underset{a' \in \mathcal{A}}{\arg\max}\; Q_k(s, a')\right)$$

For example, ε-greedy is GLIE if ε_k reduces to zero at $\varepsilon_k = \frac{1}{k}$

- i.e. decay ε over time according to a *hyperbolic schedule*

Note that the term $\mathbf{1}(S_t = s)$ is an *indicator function* that equals 1 if the condition inside is true, and 0 otherwise.

$$\mathbf{1}(S_t = s) = \begin{cases} 1, & \text{if } S_t = s \\ 0, & \text{otherwise} \end{cases}$$

It acts as a *selector* that ensures the update is applied only to the state currently being visited.

- The boldface notation $\mathbf{1}(S_t = s)$ simply emphasises that this is a function, not a constant.

GLIE Monte-Carlo Control

Sample kth episode using π: $\{S_1, A_1, R_2, \ldots, S_T\} \sim \pi$

For each state S_t and action A_t in the episode update an incremental mean,

$$
\begin{aligned}
N(S_t, A_t) &\leftarrow N(S_t, A_t) + 1 \\
Q(S_t, A_t) &\leftarrow Q(S_t, A_t) + \frac{1}{N(S_t, A_t)}\Big(G_t - Q(S_t, A_t)\Big)
\end{aligned}
$$

Improve policy based on new action–value function, replacing Q values at each step

$$
\begin{aligned}
\varepsilon &\leftarrow \frac{1}{k} \\
\pi &\leftarrow \varepsilon\text{-greedy}(Q)
\end{aligned}
$$

- In practice don't need to store π, just store Q (π becomes implicit)

Theorem

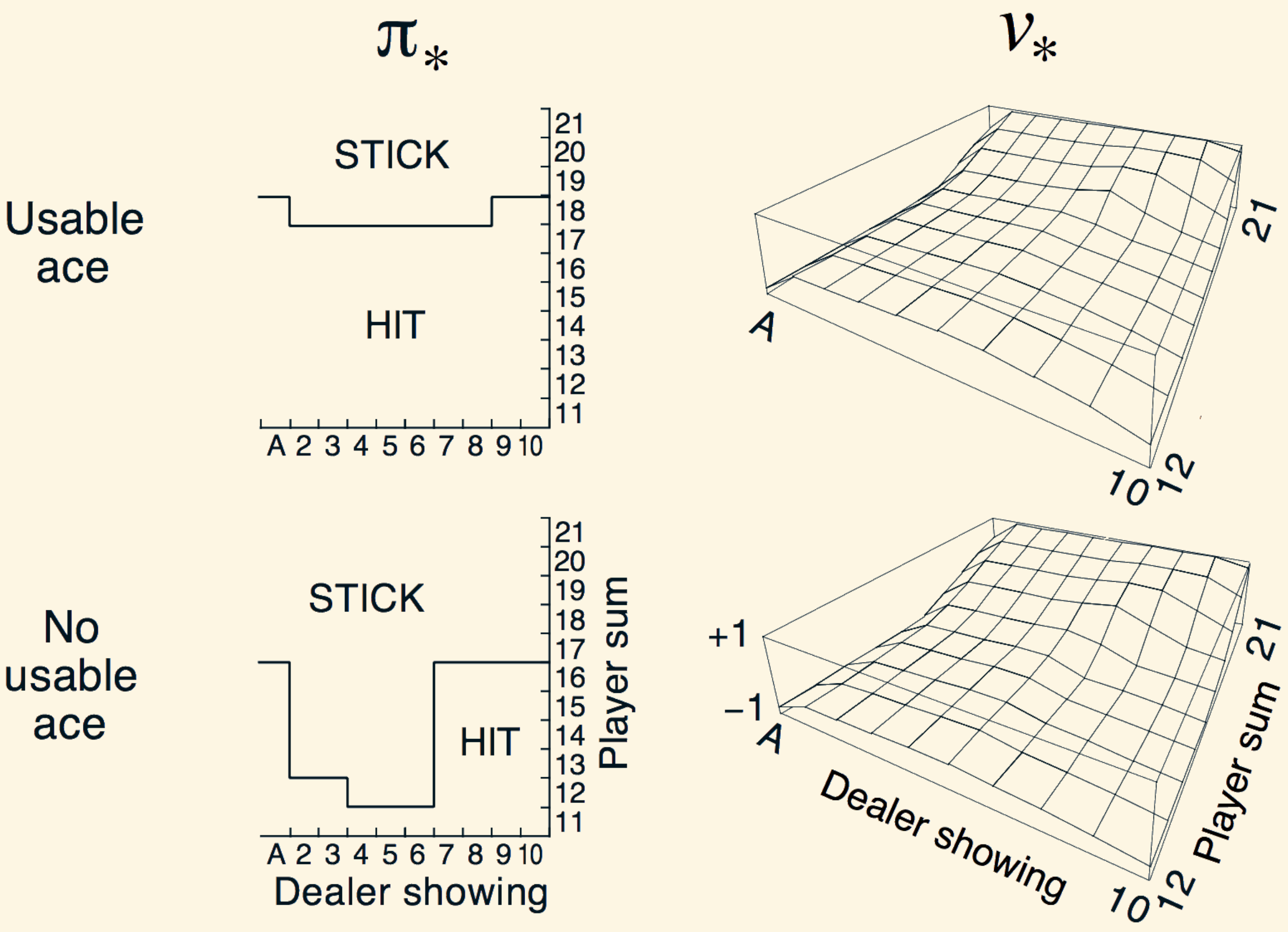
GLIE Monte Carlo control converges to the optimal action–value function,

$$Q(s,a) \quad \rightarrow \quad q_*(s,a)$$

- Converges to the optimal policy, π_*
- Every episode Monte-Carlo is substantially more efficient than running multiple episodes at each step

Back to the Blackjack Example

Monte-Carlo Control in Blackjack



Monte-Carlo Control algorithm finds the optimal policy!

(Note: *Stick* is equivalent to *hold* in this Figure)

On-Policy Temporal-Difference Learning

MC versus TD Control (Gain efficiency by Bootstrapping)

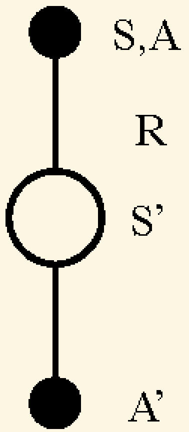
Temporal-difference (TD) learning has several advantages over Monte-Carlo (MC)

- Lower variance
- Online (including non-terminating)
- Incomplete sequences

Natural idea: use TD instead of MC in our control loop

- Apply TD to $Q(S, A)$
- Use ε-greedy policy improvement
- Update *every* time-step

Updating Action-Value Functions with Sarsa

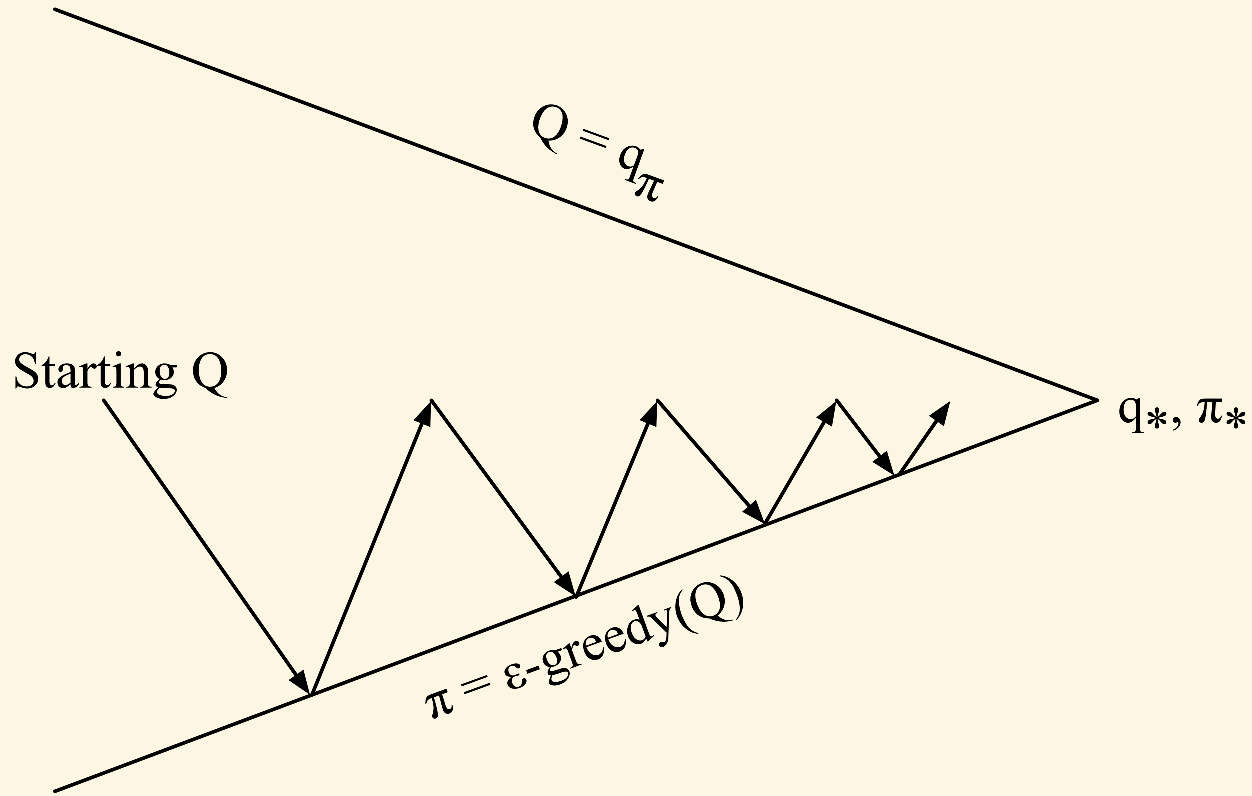


$$Q(S, A) \;\leftarrow\; Q(S, A) + \alpha\Big(R + \gamma Q(S', A') - Q(S, A)\Big)$$

Starting in state-action pair S, A, sample reward R from environment, then sample our own policy in S' for A' (note S' is chosen by the environment)

- Moves $Q(S, A)$ value in direction of *TD Target* - $Q(S, A)$ (as in Bellman equation for Q).

On-Policy Control with Sarsa



Every time-step:

Policy evaluation Sarsa, $Q \approx q_\pi$

Policy improvement $\varepsilon-$Greedy policy improvement

Sarsa Algorithm for On-Policy Control

Sarsa (On-Policy)

Initialise $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily except that $Q(\text{terminal-state}, \cdot) = 0$

Loop for each episode:

  Initialise S

  Choose A from S using policy derived from Q (e.g., ε-greedy)

  Loop for each step of episode:

    Take action A, observe R, S' (environment takes us to state S')

    Choose A' from S' using policy derived from Q (e.g., ε-greedy)

    $Q(S,A) \;\leftarrow\; Q(S,A) + \alpha\big[R + \gamma Q(S',A') - Q(S,A)\big]$

    $S \;\leftarrow\; S'; A \;\leftarrow\; A'$

until S is terminal

RHS of $Q(S,A)$ update is on-policy version of Bellman equation—expectation of what happens in environment to state S' and what happens under our own policy from that state S' onwards.

Convergence of Sarsa (Stochastic optimisation theory)

Theorem

Sarsa converges to the optimal action-value function,
$Q(s, a) \rightarrow q_*(s, a)$*, under the following conditions:*

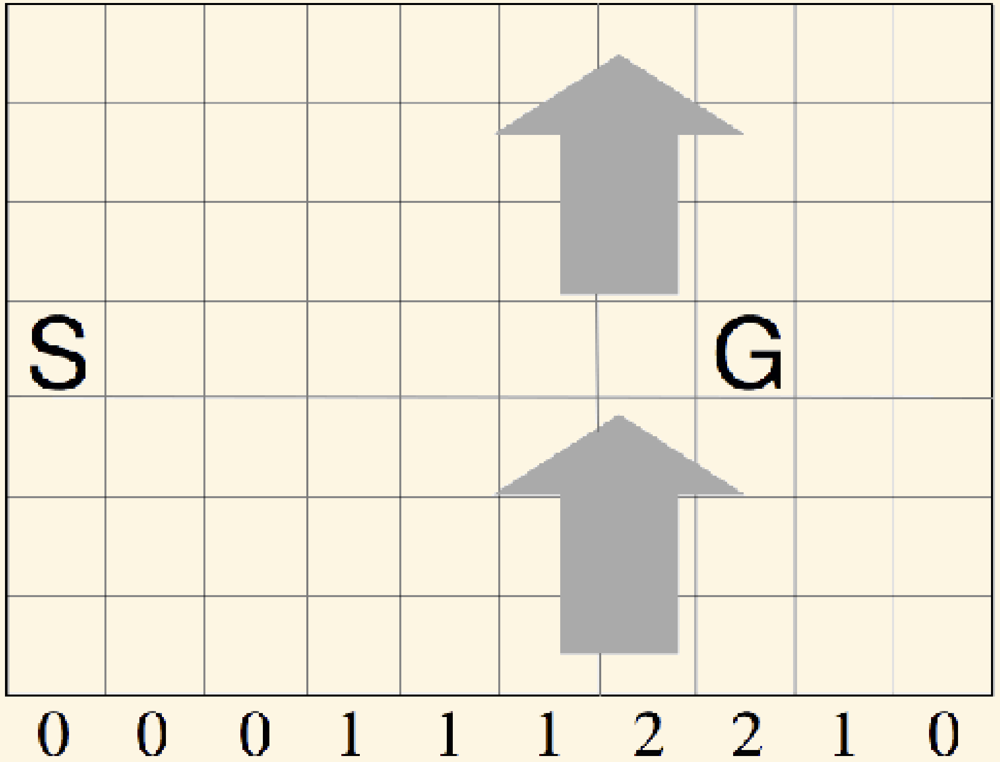
- GLIE sequence of policies $\pi_t(a \mid s)$
- Robbins-Monro sequence of step-sizes α_t

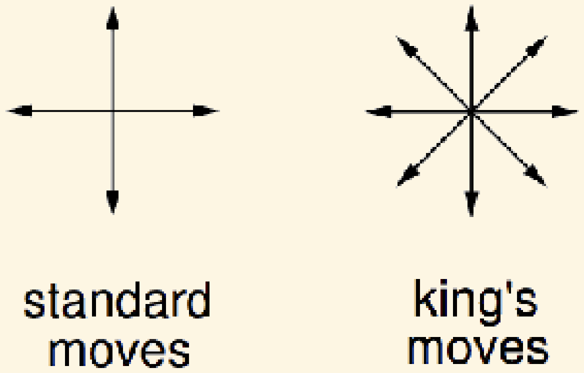
$$\sum_{t=1}^{\infty} \alpha_t = \infty$$

$$\sum_{t=1}^{\infty} \alpha_t^2 < \infty$$

Tells us that step sizes must be sufficiently large to move us as far as you want; and changes to step sizes must result in step eventually vanishing

Windy Gridworld Example



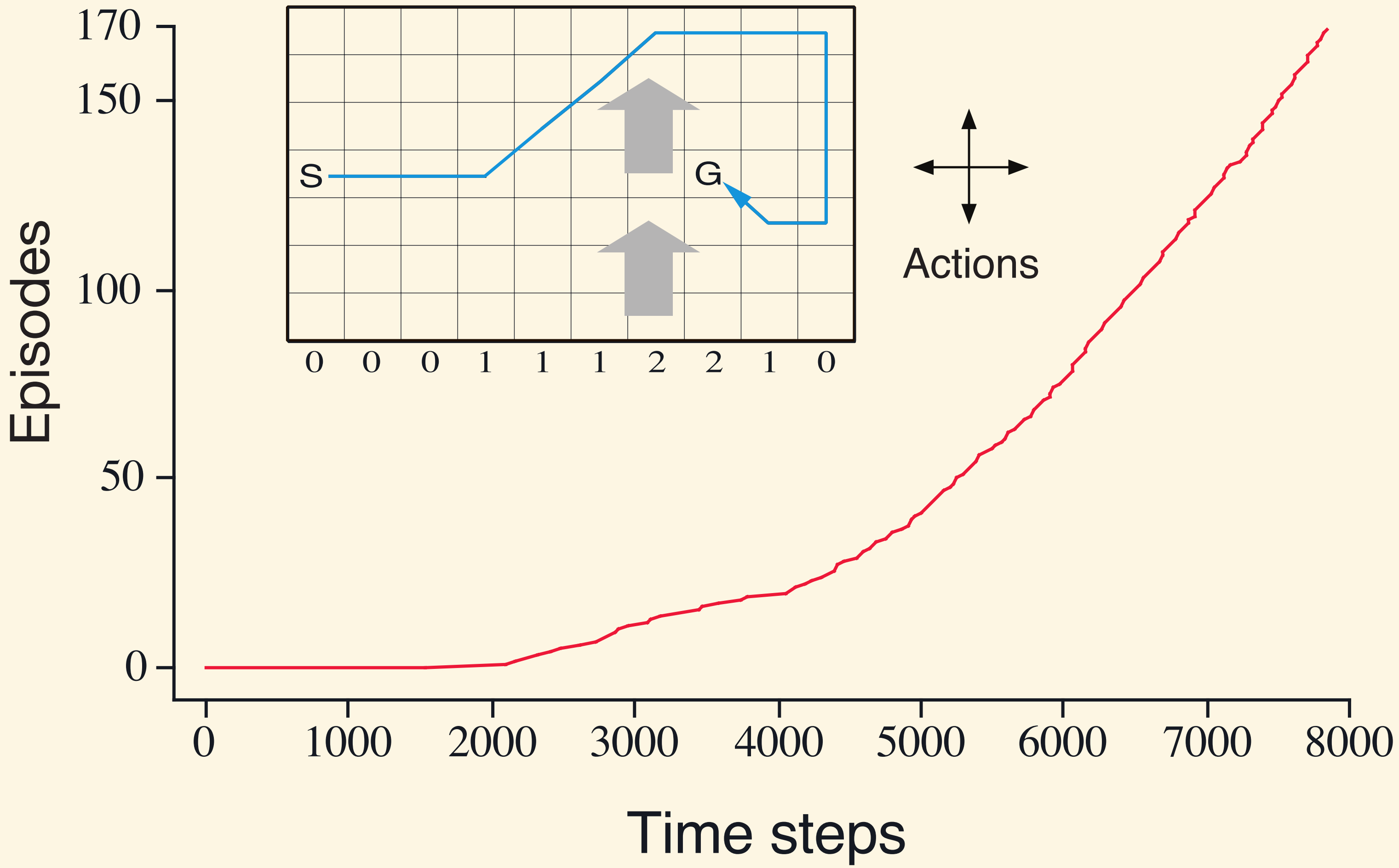


Numbers under each column is how far you get blown up per time step

- Reward $= -1$ per time-step until reaching goal

(Undiscounted and uses fixed step size α in this example)

Sarsa on the Windy Gridworld



Episodes completed (vertical axis) versus time steps (horizontal axis)

n-Step Sarsa (Bias-variance trade-off)

Consider the following n-step returns for $n = 1, 2, \infty$:

$$
\begin{array}{llll}
n = 1 & \text{(Sarsa)} & q_t^{(1)} & = R_{t+1} + \gamma Q(S_{t+1}) \\
n = 2 & & q_t^{(2)} & = R_{t+1} + \gamma R_{t+2} + \gamma^2 Q(S_{t+2}) \\
\vdots & & \vdots & \\
n = \infty & \text{(MC)} & q_t^{(\infty)} & = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{T-1} R_T
\end{array}
$$

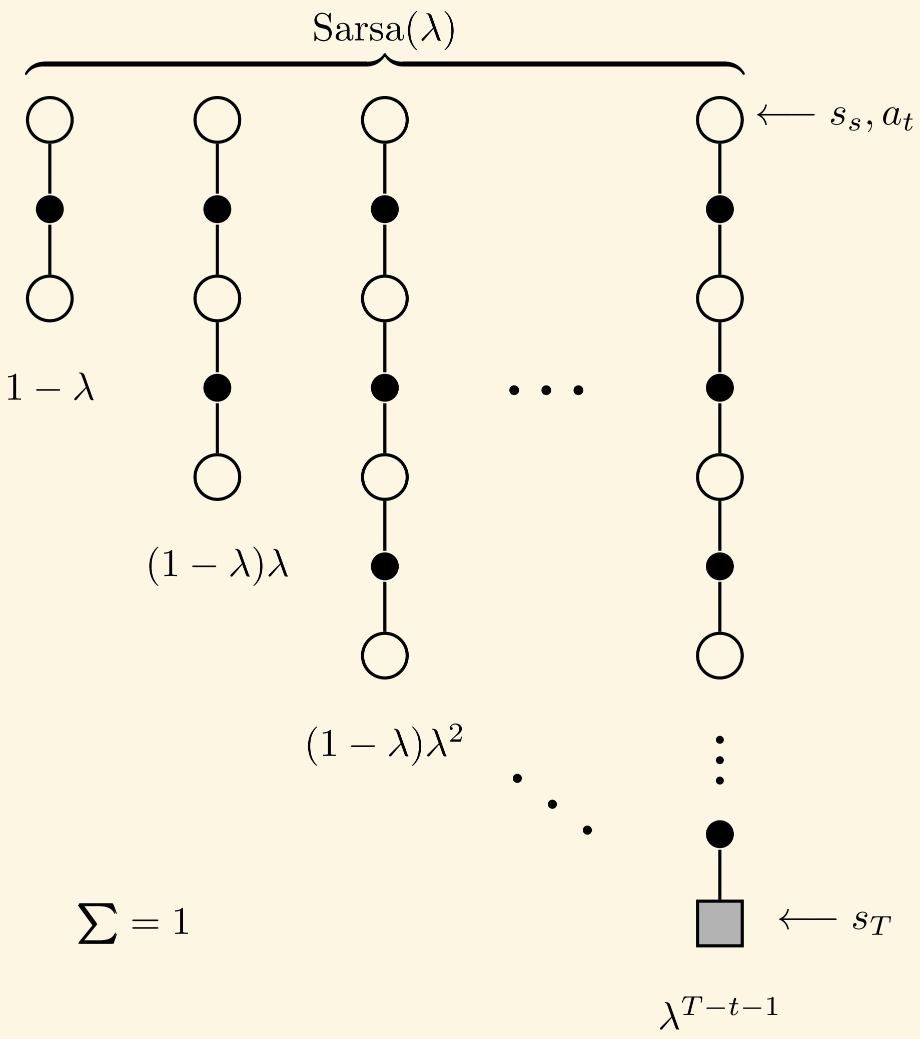
Define the n-step Q-return:

$$q_t^{(n)} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n Q(S_{t+n})$$

n-step Sarsa updates $Q(s, a)$ towards the n-step Q-return:

- $n = 1$: high bias, low variance
- $n = \infty$: no bias, high variance

Forward View Sarsa(λ)



We can do the same thing for control as we did in model-free prediction:

The q^λ *return* combines all n-step Q-returns $q_t^{(n)}$

Using weight $(1-\lambda)\lambda^{n-1}$:

$$q_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} q_t^{(n)}$$

Forward-view Sarsa(λ):

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\left(q_t^\lambda - Q(S_t, A_t)\right)$$

Backward View Sarsa(λ)

Just like TD(λ), we use eligibility traces in an online algorithm

- However Sarsa(λ) has one eligibility trace for each *state–action pair*

$$
\begin{aligned}
E_0(s, a) &= 0 \\
E_t(s, a) &= \gamma \lambda E_{t-1}(s, a) + \mathbf{1}(S_t = s, A_t = a)
\end{aligned}
$$

$Q(s, a)$ is updated for every state s and action a

- In proportion to TD-error δ_t and eligibility trace $E_t(s, a)$

$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)$$

$$Q(s, a) \leftarrow Q(s, a) + \alpha \, \delta_t \, E_t(s, a)$$

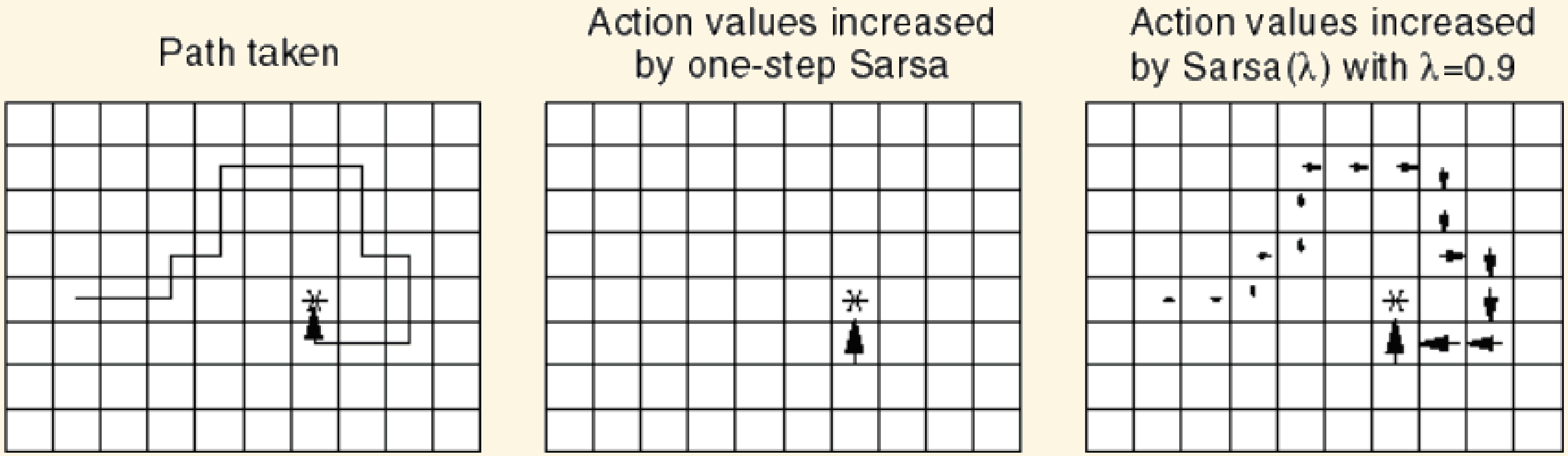
Sarsa(λ) Algorithm

Sarsa(λ)

Initialise $Q(s, a)$ arbitrarily, $\forall s \in \mathcal{S}, a \in \mathcal{A}(s)$
Loop for each episode:
  $E(s, a) = 0$, for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$
  Initialise S, A
  Loop for each step of episode:
    Take action A, observe R, S'
    Choose A' from S' using policy derived from Q (e.g., ε-greedy)
    $\delta \leftarrow R + \gamma Q(S', A') - Q(S, A)$
    $E(S, A) \leftarrow E(S, A) + 1$
    For all $s \in \mathcal{S}, a \in \mathcal{A}(s)$:
      $Q(s, a) \leftarrow Q(s, a) + \alpha\, \delta\, E(s, a)$
      $E(s, a) \leftarrow \gamma \lambda E(s, a)$
    $S \leftarrow S'$; $A \leftarrow A'$
until S is terminal

Algorithm updates *all* action-state pairs $Q(s, a)$ at *each* step of an episode

Sarsa(λ) Gridworld Example



Sarsa(0)

Sarsa(λ)

- Assume initialise action-values to zero
- Size of arrow indicates magnitude of $Q(A, S)$ value for that state
- Sarsa updates *all* action-state pairs $Q(s, a)$ at *each* step of episode

Off-Policy Learning

Off-Policy Learning

Evaluate target policy $\pi(a \mid s)$ to compute $v_\pi(s)$ or $q_\pi(s, a)$
while following a *behaviour* policy $\mu(a \mid s)$

$$\{S_1, A_1, R_2, \ldots, S_T\} \sim \mu$$

Why is this important?

- Learning from observing humans or other agents (either AI or simulated)
- Re-use experience generated from old policies $\pi_1, \pi_2, \ldots, \pi_{t-1}$
- Learn about *optimal* policy while following *exploratory* policy
- Learn about *multiple* policies while following *one* policy

Importance Sampling

Estimate the expectation of a different distribution

$$
\begin{aligned}
\mathbb{E}_{X \sim P}[f(X)] &= \sum P(X) f(X) \\
&= \sum Q(X) \frac{P(X)}{Q(X)} f(X) \\
&= \mathbb{E}_{X \sim Q}\left[\frac{P(X)}{Q(X)} f(X)\right]
\end{aligned}
$$

A technique for estimating expectations by sampling from different distributions

- Re-weights by dividing and multiplying samples to correct mismatch between expectations of the different distributions

Importance Sampling for Off-Policy Monte-Carlo

Use returns generated from μ to evaluate π

- Weight return G_t according to similarity between policies
- Multiply importance sampling corrections along entire episode

$$G_t^{\pi/\mu} = \frac{\pi(A_t \mid S_t)}{\mu(A_t \mid S_t)} \frac{\pi(A_{t+1} \mid S_{t+1})}{\mu(A_{t+1} \mid S_{t+1})} \cdots \frac{\pi(A_T \mid S_T)}{\mu(A_T \mid S_T)} G_t$$

Updates values towards *corrected* return

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t^{\pi/\mu} - V(S_t) \right)$$

- Cannot use if μ is zero when π is non-zero
- However, importance sampling *dramatically* increases variance, in case of Monte-Carlo learning

Importance Sampling for Off-Policy TD

Use TD targets generated from μ to evaluate π

- Weight TD target $R + \gamma V(S')$ by importance sampling

Only need a single importance sampling correction

$$V(S_t) \leftarrow V(S_t) + \alpha\left(\frac{\pi(A_t \mid S_t)}{\mu(A_t \mid S_t)}\left(R_{t+1} + \gamma V(S_{t+1})\right) - V(S_t)\right)$$

- Much lower variance than Monte Carlo importance sampling (policies only need to be similar over a *single* step)
- In practice you have to use TD learning when working off-policy (it becomes imperative to bootstrap)

Q-Learning

We now consider off-policy learning of action-values, $Q(s, a)$

No importance sampling is required using action-values as you can bootstrap as follows

- Next action is chosen using *behaviour* policy $A_{t+1} \sim \mu(\cdot \mid S_t)$
- But we consider *alternative* successor action $A' \sim \pi(\cdot \mid S_t)$
- $\cdots$ and we update $Q(S_t, A_t)$ towards value of alternative action

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\Big(R_{t+1} + \gamma Q(S_{t+1}, A') - Q(S_t, A_t)\Big)$$

Q-learning is the technique that works best with off-policy learning

Off-Policy Control with Q-Learning

We can now allow *both* behaviour and target policies to improve

- The target policy π is greedy w.r.t. $Q(s, a)$

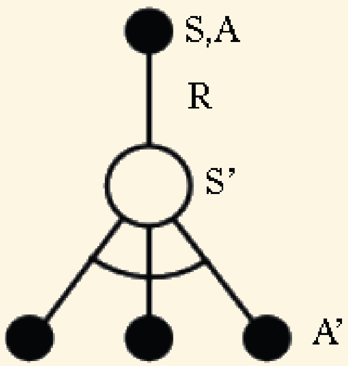
$$\pi(S_{t+1}) = \arg\max_{a'} Q(S_{t+1}, a')$$

- The behaviour policy μ is e.g. ε-greedy w.r.t. $Q(s, a)$

The Q-learning target then simplifies according to:

$$\begin{aligned} & R_{t+1} + \gamma Q(S_{t+1}, A') \\ = & R_{t+1} + \gamma Q\left(S_{t+1}, \arg\max_{a'} Q(S_{t+1}, a')\right) \\ = & R_{t+1} + \max_{a'} \gamma Q(S_{t+1}, a') \end{aligned}$$

Q-Learning Control Algorithm



$$Q(S,A) \leftarrow Q(S,A) + \alpha\left(R + \gamma \max_{a'} Q(S',a') - Q(S,A)\right)$$

Theorem

Q-learning control converges to the optimal action-value function, $Q(s,a) \rightarrow q_*(s,a)$

Q-Learning Algorithm for Off-Policy Control

$Q-$Learning (Off-Policy)

Initialise $Q(s, a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\text{terminal-state}, \cdot) = 0$

Loop for each episode:

  Initialise S

  Loop for each step of episode:

    Choose A from S using policy derived from Q (e.g., ε-greedy)

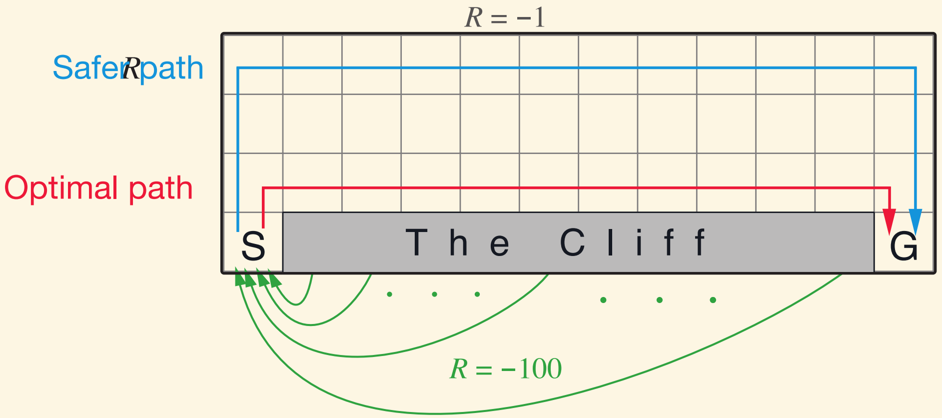
    Take action A, observe R, S'

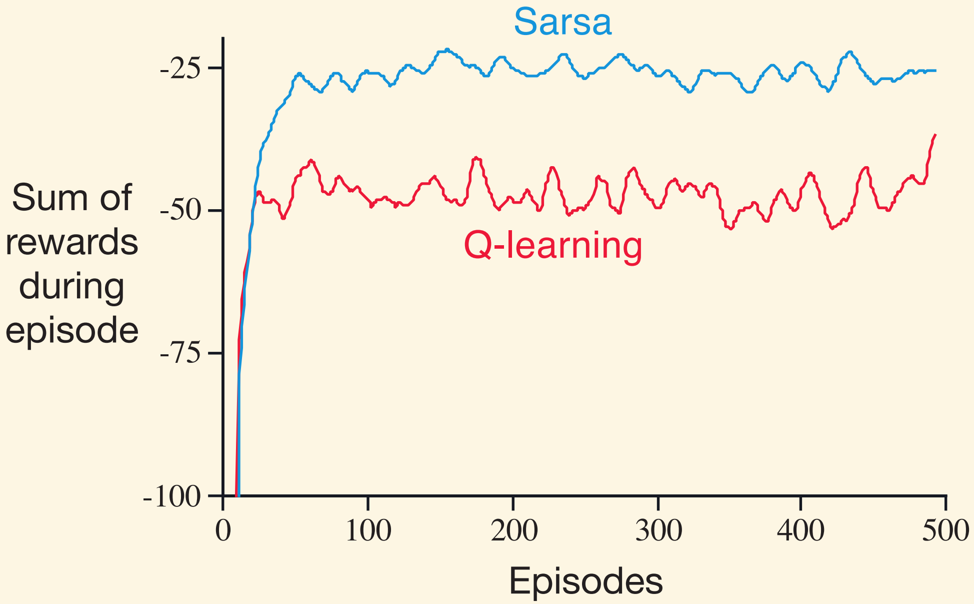
    $Q(S, A) \ \leftarrow \ Q(S, A) + \alpha \Big[R + \gamma \max_a Q(S', a) - Q(S, A) \Big]$

    $S \ \leftarrow \ S'$

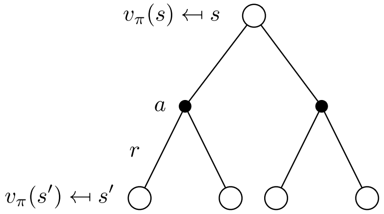
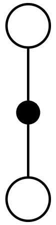
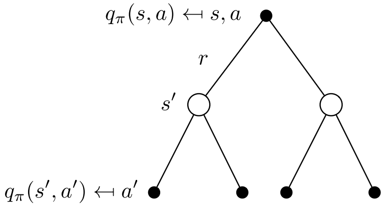
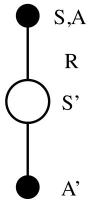
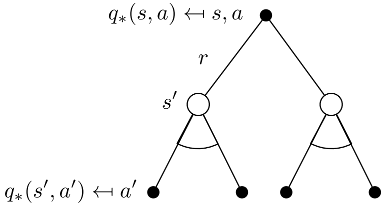
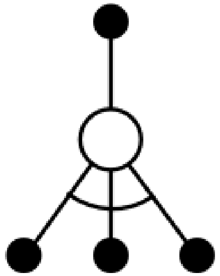
until S is terminal

Cliff Walking Example (Sarsa versus Q-Learning)





Relationship between Tree Backup and TD

	Full Backup (DP)	*Sample Backup (TD)*
Bellman Expectation Equation for $v_\pi(s)$	 Iterative Policy Evaluation	 TD Learning
Bellman Expectation Equation for $q_\pi(s, a)$	 Q-Policy Iteration	 Sarsa
Bellman Optimality Equation for $q_*(s, a)$	 Q-Value Iteration	 Q-Learning

Relationship between DP and TD

Dynamic Programming (DP)	*Sample Backup (TD)*
Iterative Policy Evaluation $V(s) \leftarrow \mathbb{E}[R + \gamma V(S') \mid s]$	**TD Learning** $V(S) \xleftarrow{\alpha} R + \gamma V(S')$
Q-Policy Iteration $Q(s,a) \leftarrow \mathbb{E}[R + \gamma Q(S', A') \mid s, a]$	**Sarsa** $Q(S,A) \xleftarrow{\alpha} R + \gamma Q(S', A')$
Q-Value Iteration $Q(s,a) \leftarrow \mathbb{E}[R + \gamma \max_{a' \in \mathcal{A}} Q(S', a') \mid s, a]$	**Q-Learning** $Q(S,A) \xleftarrow{\alpha} R + \gamma \max_{a' \in \mathcal{A}} Q(S', a')$

where $x \xleftarrow{\alpha} y \equiv x \leftarrow x + \alpha (y - x)$